Chapter 6

Riesz Representation Theorems

6.1 Dual Spaces

Definition 6.1.1. Let $V$ and $W$ be vector spaces over $\mathbb{R}$. We let

$$L(V,W) = \{ T : V \to W \mid T \text{ is linear} \}.$$ 

The space $L(V,\mathbb{R})$ is denoted by $V^\ast$ and elements of $V^\ast$ are called linear functionals.

Example 6.1.2. 1) Let $V = \mathbb{R}^n$. Then we can identify $\mathbb{R}^\ast$ with $\mathbb{R}$ as follows:

For each $a = (a_1, a_2, \ldots, a_n)$ define $\phi_a : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi_a((x_1, x_2, \ldots, x_n)) = x \cdot a = \sum_{i=1}^n x_i a_i.$$ 

2) Let $(X,d)$ be a compact metric space. Let $x_0 \in X$. Define $\phi_{x_0} : C(X) \to \mathbb{R}$ by

$$\phi_{x_0}(f) = f(x_0).$$ 

Then $\phi_{x_0} \in C(X)^\ast$.

Definition 6.1.3. Let $(V, \| \cdot \|_V)$ and $(W, \| \cdot \|_W)$ be normed linear spaces. Let $T : V \to W$ be linear. We say that $T$ is bounded is

$$\sup_{\|x\|_V \leq 1} \{ \| T(x) \|_W \} < \infty.$$ 

In this case, we write

$$\| T \| = \sup_{\|x\|_V \leq 1} \{ \| T(x) \|_W \}.$$ 

Otherwise, we say that $T$ is unbounded.

The next result establishes the fundamental criterion for when a linear map between normed linear spaces is continuous. It’s proof is left as an exercise.

Theorem 6.1.4. Let $(V, \| \cdot \|_V)$ and $(W, \| \cdot \|_W)$ be normed linear spaces. Let $T : V \to W$ be linear. Then the following are equivalent.

1) $T$ is continuous.

2) $T$ is continuous at 0.
2) \( T \) is bounded.

Proof. 1) \( \Rightarrow \) 2) This is immediate.

2) \( \Rightarrow \) 3) Assume that \( T \) is continuous at 0. Let \( \delta \) be such that if \( \| x \| \leq \delta \), then \( \| T(x) \| \leq W \). It follows easily that \( \| T \| \leq \frac{1}{\delta} \).

3) \( \Rightarrow \) 1) Note that we may assume that \( \| T \| > 0 \) otherwise \( T = 0 \) and hence is obviously continuous.

Let \( x_0 \in V \) and let \( \epsilon > 0 \). Let \( \delta = \frac{\epsilon}{\| T \|} \). Then if \( \| x - x_0 \| < \delta \), we have

\[
\| T(x) - T(x_0) \| = \| T(x - x_0) \| \leq \| T \| \cdot \| x - x_0 \| < \epsilon.
\]

Remark 6.1.5. 1) Let \((V, \| \cdot \|_V)\) and \((W, \| \cdot \|_W)\) be normed linear spaces. Let \( T : V \to W \) be linear.

Then we can easily deduce from the previous theorem that \( \) \( \) if \( T \) is bounded, then \( T \) is uniformly continuous.

2) Let

\[
B(V, W) = \{ T : X \to Y | T \, \text{is linear and} \, T \, \text{is bounded} \}.
\]

Let \( T_1 \) and \( T_2 \) be in \( B(V, W) \). Then if \( x \in V \), we have

\[
\| T_1 + T_2(x) \|_W = \| T_1(x) + T_2(x) \|_W \\
\leq \| T_1(x) \|_W + \| T_2(x) \|_W \\
\leq \| T_1 \| \cdot \| x \|_V + \| T_1 \| \cdot \| x \|_V \\
= (\| T_1 \| + \| T_2 \|) \cdot \| x \|_V.
\]

As such \( T_1 + T_2 \in B(V, W) \) and in particular

\[
\| T_1 + T_2 \| \leq \| T_1 \| + \| T_2 \|.
\]

It follows that \((B(V, W), \| \cdot \|)\) is also a normed linear space.

Theorem 6.1.6. Assume that \((W, \| \cdot \|_W)\) be a Banach space. Then so is \((B(V, W), \| \cdot \|)\).

Proof. Assume that \( \{ T_n \} \) is Cauchy. Let \( x \in V \). Since

\[
\| T_n(x) - T_m(x) \|_W \leq \| T_n - T_m \| \cdot \| x \|_V
\]

it follows easily that \( \{ T_n(x) \} \) is also Cauchy in \( W \). As such we can define \( T_0 \) by

\[
T_0(x) = \lim_{n \to \infty} T_n(x).
\]

To see that \( T_0 \) is linear observe that

\[
T_0(\alpha x + \beta y) = \lim_{n \to \infty} T_n(\alpha x + \beta y) \\
= \lim_{n \to \infty} \alpha T_n(x) + \beta T_n(y) \\
= \alpha T_0(x) + \beta T_0(y).
\]

To see that \( T_0 \) is bounded first observe that being Cauchy, \( \{ T_n \} \) is bounded. Hence we can find an \( M > 0 \) such that \( \| T_n \| \leq M \) for each \( n \in \mathbb{N} \). Moreover, since \( \| T_0(x) \|_W = \lim_{n \to \infty} \| T_n(x) \| \leq M \| x \|_V \), we have that \( \| T_0 \| \leq M \).
Now let $\epsilon > 0$ and choose an $N \in \mathbb{N}$ so that if $n, m \geq N$, then
\[ \|T_n - T_m\| < \epsilon. \]
Let $x \in V$ with $\|x\|_V \leq 1$. Then since $\|T_n(x) - T_m(x)\|_W < \epsilon$ for each $m \geq N$, we have
\[ \|T_n(x) - T_0(x)\|_W = \lim_{m \to \infty} \|T_n(x) - T_m(x)\|_W \leq \epsilon. \]
In particular
\[ T_0 = \lim_{n \to \infty} T_n \]
in $B(X,Y)$.

**Definition 6.1.7.** Let $(V, \| \cdot \|)$ be a normed linear space. The space $B(V, \mathbb{R})$ is called the dual space of $V$ and is denoted by $V^*$.

**Example 6.1.8.** 1) Let $V = \mathbb{R}^n$ with the usual norm $\| \cdot \|_2$. For each $a = (a_1, a_2, \ldots, a_n)$ we defined $\phi_a : \mathbb{R}^n \to \mathbb{R}$ by
\[ \phi_a((x_1, x_2, \ldots, x_n)) = x \cdot a = \sum_{i=1}^n x_i a_i. \]
Then in fact $\phi_a \in \mathbb{R}^n^*$ and
\[ \|\phi_a\| = \|a\|_2. \]
2) Let $(X, d)$ be a compact metric space. Again, if $x_0 \in X$ and we define $\phi_{x_0} : (C(X), \| \cdot \|_\infty) \to \mathbb{R}$ by
\[ \phi_{x_0}(f) = f(x_0), \]
then $\phi_{x_0} \in C(X)^*$. In this case $\|\phi_{x_0}\|$. 3) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Hölder’s Inequality allows us to define for each $g \in L_q(X, \mathcal{A}, \mu)$ and element $\phi_g \in L_p(X, \mathcal{A}, \mu)^\sharp$ by
\[ \phi_g(f) = \int fg \, d\mu. \]
Moreover, Hölder’s Inequality also shows that $\phi_g \in L_p(X, \mathcal{A}, \mu)^*$ with
\[ \|\phi_g\| \leq \|g\|_p. \]
Note that $\phi_g$ has the additional property that if $g \geq 0 \mu$-a.e., then $\phi_g(f) \geq 0$ whenever $f \in L_p(X, \mathcal{A}, \mu)$ and $f \geq 0 \mu$-a.e.

4) Let $(X, d)$ be a compact measure space and let $\mu$ be a finite regular signed measure on $\mathcal{B}(X)$. Define $\phi_\mu \in C(X)^\sharp$ by
\[ \phi_\mu(f) = \int f \, d\mu. \]
Since
\[ |\phi_\mu(f)| \leq \int |f| \, d|\mu| \leq \|f\|_\infty \|\mu\|_{\text{meas}} \]
we see that in fact $\phi_\mu \in C(X)^*$ and $\|\phi_\mu\| \leq \|\mu\|_{\text{meas}}$. 93
We note again that $\phi_\mu$ has the additional property that if $\mu$ is a positive measure on $\mathcal{B}(X)$, then $\phi_\mu(f) \geq 0$ whenever $f \in C(X)$ and $f \geq 0$. Furthermore, in this case since

$$\phi_\mu(1) = \int 1 \, d\mu = \mu(X) = \|\mu\|_{\text{meas}},$$

if $\mu$ is a positive measure we have

$$\|\phi_\mu\| = \|\mu\|_{\text{meas}}.$$

**Problem 6.1.9.** In Examples 3) and 4) above we have shown respectively that every element in $L_\varphi(X,\mathcal{A},\mu)$ determines a continuous functional on $L_p(X,\mathcal{A},\mu)$ and that if $(X,d)$ is a compact metric space, then every finite regular signed measure on $\mathcal{B}(X)$ determines a continuous linear functional on $C(X)$. It is natural to ask:

Do all continuous linear functionals on $L_p(X,\mathcal{A},\mu)$ and $C(X)$ arise in this fashion?

### 6.2 Riesz Representation Theorem for $L^p(X,\mathcal{A},\mu)$

In this section we will focus on the following problem:

**Problem 6.2.1.** What is $L^p(X,\mathcal{A},\mu)^*$?

We have already established most of the following result:

**Lemma 6.2.2.** If $(X,\mathcal{A},\mu)$ is a measure space and if $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for every $g \in L^q(X,\mu)$ the map $\Gamma_g : L^p(X,\mu) \rightarrow \mathbb{R}$ defined by $\Gamma_g(f) = \int_X f \, g \, d\mu$ is a continuous linear functional on $L^p(X,\mu)$. Further, $\|\Gamma_g\| \leq \|g\|_q$ and if $1 < p \leq \infty$ then $\|\Gamma_g\| = \|g\|_q$.

**Proof.** Assignment.

If $(X,\mu)$ is $\sigma$-finite, then equality holds for $p = 1$ as well.

**Lemma 6.2.3.** Let $(X,\mathcal{A},\mu)$ be a finite measure space and if $1 \leq p < \infty$. Let $g$ be an integrable function such that there exists a constant $M$ with $\int |f| \, g \, d\mu \leq M \|\varphi\|_p$ for all simple functions $\varphi$. Then $g \in L^q(X,\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** Assume that $p > 1$. Let $\psi_n$ be a sequence of simple functions with $\psi_n \nearrow |g|^q$. Let $\varphi_n = (\psi_n)^\frac{1}{q} \text{sgn}(g)$. Then $\varphi_n$ is also simple and $\|\varphi_n\|_p = (\int \psi_n \, d\mu)^\frac{1}{p}$. Since $|\varphi_n g| \geq |\varphi_n| |\psi_n| \frac{1}{q} = |\psi_n|$, we have

$$\int \psi_n \, d\mu \leq \int \varphi_n g \, d\mu \leq M \|\varphi_n\|_p = M \left(\int \psi_n \, d\mu\right)^\frac{1}{p}$$

Therefore $\int \psi_n \, d\mu \leq M^q$. By the Monotone Convergence Theorem we get that $\|g\|_q \leq M$, so $g \in L^q(X,\mu)$. If $p = 1$, then we need to show that $g$ is bounded almost everywhere. Let $E = \{x \in X \mid |g(x)| > M\}$. Let $f = \frac{1}{\mu(E)} \chi_E \text{sgn}(g)$. Then $f$ is a simple function and $\|f\|_1 = 1$. This is a contradiction.

**Lemma 6.2.4.** Let $1 \leq p < \infty$. Let $\{E_n\}$ be a sequence of disjoint sets. Let $\{f_n\} \subseteq L^p(X,\mu)$ be such that $f_n(x) = 0$ if $x \notin E_n$ for each $n \geq 1$. Let $f = \sum_{n=1}^{\infty} f_n$. Then $f \in L^p(X,\mu)$ if and only if $\sum_{n=1}^{\infty} \|f_n\|_p^p < \infty$.

**Proof.** Exercise.
Theorem 6.2.5 [Riesz Representation Theorem, I]. Let $\Gamma \in L^p(X, \mu)^*$, where $1 \leq p < \infty$ and $\mu$ is $\sigma$-finite. Then if $\frac{1}{p} + \frac{1}{q} = 1$, there exists a unique $g \in L^q(X, \mu)^*$ such that

$$\Gamma(f) = \int_X fg \, d\mu = \phi_g(f)$$

Moreover, $\|\Gamma\| = \|g\|_q$.

Proof. Assume that $\mu$ is finite. Then every bounded measurable function is in $L^p(X, \mu)$. Define $\lambda : A \to \mathbb{R} : E \mapsto \Gamma(\chi_E)$. Let $\{E_n\} \subseteq A$ be a sequence of disjoint sets, and let $E = \bigcup_{n=1}^{\infty} E_n$. Let $\alpha_n = \text{sgn}(\chi_{E_n})$ and $f = \sum_{n=1}^{\infty} \alpha_n \chi_{E_n}$. Then $f \in L^p(X, \mu)$ and $\Gamma(f) = \sum_{n=1}^{\infty} |\lambda(E_n)| < \infty$ and so $\sum_{n=1}^{\infty} |\lambda(E_n)| = \Gamma(\chi_E) = \lambda(E)$. Therefore $\lambda$ is a finite signed measure. Clearly, if $\mu(E) = 0$ then $\chi_E = 0$ almost everywhere, so $\lambda(E) = \Gamma(0) = 0$. Therefore $|\lambda| \ll \mu$. By the Radon-Nikodym Theorem, there is an integrable function $g$ such that $\lambda(E) = \int_E g \, d\mu$ for all $E \in A$. If $\varphi$ is simple, then $\Gamma(\varphi) = \int \varphi \, d\mu$ by linearity of the integral. But $|\Gamma(\varphi)| \leq \|\Gamma\| \|\varphi\|_p$ for all simple functions $\varphi$, so $g \in L^q(X, \mu)$ by the lemma above. Now $\Gamma - \phi_g \in L^p(X, \mu)^*$ and $\Gamma - \phi_g = 0$ on the space of simple functions. Since the simple functions are dense in $L^p(X, \mu)$, $\Gamma - \phi_g = 0$ on $L^p(X, \mu)$, so $\Gamma = \phi_g$. We have that $\|\Gamma\| = \|\phi_g\| = \|g\|_q$, as before.

Now assume that $\mu$ is $\sigma$-finite. We can write $X = \bigcup_{n=1}^{\infty} X_n$, where $\mu(X_n) < \infty$ and $X_n \subseteq X_{n+1}$ for all $n \geq 1$. For each $n \geq 1$, the proof above gives us $g_n \in L^q(X_n, \mu)$, vanishing outside $X_n$, such that $\Gamma(f) = \int f \, g \, d\mu$ for all $f \in L^p(X, \mu)$ vanishing off of $X_n$. Moreover, $\|g_n\|_q \leq \|\Gamma\|$. By the uniqueness of the $g_n$’s, we can assume that $g_{n+1} = g_n$ on $X_n$. Let $g(x) = \lim_{n \to \infty} g_n(x)$. We have that $|g_n| \nearrow |g|$. By the Monotone Convergence Theorem

$$\int |g| \, d\mu = \lim_{n \to \infty} \int |g_n| \, d\mu \leq \|\Gamma\|q$$

Hence $g \in L^q(X, \mu)$. Let $f \in L^p(X, \mu)$ and $f_n = f \chi_{X_n}$. Then $f_n \to f$ pointwise and $f_n \in L^p(X, \mu)$ for all $n \geq 1$. Since $|fg| \in L^1(X, \mu)$ and $f_n g \leq |fg|$, the Lebesgue Dominated Convergence Theorem shows

$$\int fg \, d\mu = \lim_{n \to \infty} \int f_n g \, d\mu = \lim_{n \to \infty} \Gamma(f_n) = \Gamma(f)$$

If $p = 1$, then we cannot drop the assumption of $\sigma$-finiteness.

Theorem 6.2.6 [Riesz Representation Theorem, II]. Let $\Gamma \in L^p(X, \mu)^*$, where $1 < p < \infty$. Then if $\frac{1}{p} + \frac{1}{q} = 1$, there exists a unique $g \in L^q(X, \mu)$ such that

$$\Gamma(f) = \int f g \, d\mu$$

for all $f \in L^p(X, \mu)$. Moreover, $\|\Gamma\| = \|g\|_q$.

Proof. Let $E \subseteq X$ be $\sigma$-finite, then there exists a unique $g_E \in L^q(X, \mu)$, vanishing outside of $E$, such that $\Gamma(f) = \int f g_E \, d\mu$ for all $g \in L^p(X, \mu)$ vanishing outside of $E$. Moreover, if $A \subseteq E$, then $g_A = g_E$ almost everywhere on $A$. For each $\sigma$-finite set $E$ let $\lambda(E) = \int |g_E|^q \, d\mu$. If $A \subseteq E$, then $\lambda(A) \leq \lambda(E) \leq \|\Gamma\|^q$. Let $M = \sup\{\lambda(E) | E \text{ is } \sigma\text{-finite}\}$. Let $\{E_n\}$ be a sequence of $\sigma$-finite sets such that $\lim_{n \to \infty} \lambda(E_n) = M$. If $H = \bigcup_{n=1}^{\infty} E_n$, then $H$ is $\sigma$-finite and $\lambda(H) = M$. If $E$ is $\sigma$-finite with $H \subseteq E$, then $g_E = g_H$ almost everywhere on $H$. But

$$\int |g_E|^q \, d\mu = \lambda(E) \leq \lambda(H) = \int |g_H|^q \, d\mu$$

so $g_E = 0$ almost everywhere on $E \setminus H$. Let $g = g_H |_H$. Then $g \in L^q(X, \mu)$ and if $E$ is $\sigma$-finite with $H \subseteq E$ then $g_E = g$ almost everywhere. If $f \in L^p(X, \mu)$, then let $E = \{x \in X | f(x) \neq 0\}$. $E$ is $\sigma$-finite and hence $E_1 = E \cup H$ is $\sigma$-finite. Hence
\[ \Gamma(f) = \int f g_{E_1} \, d\mu = \int f g \, d\mu = \phi(g) \]

Therefore \( \Gamma = \phi_g \) and as before \( \|\Gamma\| = \|g\|_q \).

We have shown that if \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), then for any measure space \((X,A,\mu), L^p(X,\mu)^* \cong L^q(X,\mu)\). If \( \mu \) is \( \sigma \)-finite, then \( L^1(X,\mu)^* \cong L^\infty(X,\mu) \). What happens when \( p = \infty? \] \( L^1(X,\mu) \hookrightarrow L^\infty(X,\mu)^* \), but this embedding is not usually surjective. There exists a compact Hausdorff space \( \Omega \) such that \( L^\infty(X,\mu) \cong C(\Omega) \). What is \( C(\Omega) \)?

Let \( \varphi : [a,b] \to \mathbb{R} \) be defined by \( \varphi(f) = f(x_0) \). Then \( \varphi \in C[a,b]^* \), and \( \|\varphi\| = 1 \). Let \( \mu_{x_0} \) be the measure on \([a,b]\) of the point mass \( x_0 \). If \( g \in L^1([a,b],m) \), then \( \varphi_g(f) = \int_a^b f g \, dm \) is a linear functional on \( C[a,b] \), and \( \|\varphi_g\| \leq \|g\|_1 \). \( g \) is the Radon-Nikodym derivative of an absolutely continuous measure \( \mu \) on \([a,b]\), and \( \varphi_g(f) = \int f \, d\mu \). If \( \mu \in \text{Meas}[a,b] \), then \( \varphi_n(f) = \int f \, d\mu \) is a bounded linear functional on \( C[a,b] \), with \( \|\varphi_n\| \leq \|\mu\|_{\text{Meas}} \).

### 6.3 Riesz Representation Theorem for \( C([a,b]) \)

**Theorem 6.3.1. [Jordan Decomposition Theorem]**

Let \( \Gamma \in C([a,b])^* \). Then there exist positive linear functionals \( \Gamma^+, \Gamma^- \in C([a,b])^* \) such that

\[
\Gamma = \Gamma^+ - \Gamma^-
\]

and

\[
\|\Gamma\| = \|\Gamma^+(1)\| + \|\Gamma^-(1)\|.
\]

**Proof.** Assume that \( f \geq 0 \). Define

\[
\Gamma^+(f) = \sup_{0 \leq \phi \leq f} \Gamma(\phi).
\]

Then \( \Gamma^+(f) \geq 0 \) and \( \Gamma^+(f) \geq \Gamma(f) \). It is also easy to see that if \( c \geq 0 \), then \( \Gamma^+(cf) = c\Gamma^+(f) \).

Let \( f, g \geq 0 \). If \( 0 \leq \phi \leq f \) and \( 0 \leq \psi \leq g \), then \( 0 \leq \phi + \psi \leq f + g \) so

\[
\Gamma(\phi) + \Gamma(\psi) \leq \Gamma^+(f + g)
\]

and hence,

\[
\Gamma^+(f) + \Gamma^+(g) \leq \Gamma^+(f + g).
\]

If \( 0 \leq \psi \leq f + g \), then let \( \varphi = \inf \{f, \psi\} \) and \( \xi = \psi - \varphi \). Then \( 0 \leq \varphi \leq f \) and \( 0 \leq \xi \leq g \). It follows that

\[
\Gamma(\psi) = \Gamma(\varphi) + \Gamma(\xi) \leq \Gamma^+(f) + \Gamma^+(g).
\]

This shows that

\[
\Gamma^+(f + g) \leq \Gamma^+(f) + \Gamma^+(g)
\]

Therefore,

\[
\Gamma^+(f + g) = \Gamma^+(f) + \Gamma^+(g)
\]

Let \( f \in C[a,b] \). Let \( \alpha, \beta \) be such that \( f + \alpha 1 \geq 0 \) and \( f + \beta 1 \geq 0 \). Then

\[
\Gamma^+(f + \alpha 1 + \beta 1) = \Gamma^+(f + \alpha 1) + \Gamma^+(\beta 1) = \Gamma^+(f + \beta 1) + \Gamma^+(\alpha 1)
\]

This shows that

\[
\Gamma^+(f + \alpha 1) - \Gamma^+(\alpha 1) = \Gamma^+(f + \beta 1) - \Gamma^+(\beta 1)
\]

As such, if we let

\[
\Gamma^+(f) = \Gamma^+(f + \alpha 1) - \Gamma^+(\alpha 1),
\]

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then \( \Gamma^+ \) is well defined.

Let \( f, g \in C[a, b] \). Let \( \alpha, \beta \) be chosen so that \( f + \alpha 1 \geq 0 \) and \( g + \beta 1 \geq 0 \). Then \( f + g + (\alpha + \beta)1 \geq 0 \) so

\[
\Gamma^+(f + g) = \Gamma^+(f + g + (\alpha + \beta)1) - \Gamma^+((\alpha + \beta)1) \\
= \Gamma^+(f + \alpha 1) + \Gamma^+(g + \beta 1) - \Gamma^+((\alpha + \beta)1) \\
= \Gamma^+(f + \alpha 1) + \Gamma^+(g + \beta 1) - \Gamma^+(\beta 1) \\
= \Gamma^+(f) + \Gamma^+(g).
\]

That is \( \Gamma^+ \) is additive.

It is also clear that \( \Gamma^+(cf) = c\Gamma^+(f) \) when \( c \geq 0 \). But since \( \Gamma^+(-f) + \Gamma^+(f) = \Gamma^+(0) = 0 \), we get that

\[
\Gamma^+(-f) = -\Gamma^+(f)
\]
so \( \Gamma^+ \) is linear.

Let \( \Gamma^- = \Gamma^+ - \Gamma \)

Since it is clear that \( \Gamma^+(f) \geq \Gamma(f) \) if \( f \geq 0 \), \( \Gamma^- \) is also positive.

We know that

\[
\| \Gamma \| = \| \Gamma^+ \| + \| \Gamma^- \| = \Gamma^+(1) + \Gamma^-(1)
\]

Let \( 0 \leq \psi \leq 1 \). Then \( \| 2\psi - 1 \|_{\infty} \leq 1 \). As such

\[
\| \Gamma \| \geq \Gamma(2\psi - 1) = 2\Gamma(\psi) - \Gamma(1)
\]
and therefore

\[
\| \Gamma \| \geq 2\Gamma^+(1) - \Gamma(1) \\
= \Gamma^+(1) + \Gamma^-(1)
\]

Hence

\[
\| \Gamma \| = \Gamma^+(1) + \Gamma^-(1).
\]

\[\blacksquare\]

**Theorem 6.3.2.** (*Riesz Representation Theorem for \( C([a, b]) \)*)

Let \( \Gamma \in C([a, b])^* \). Then there exists a unique finite signed measure \( \mu \) on the Borel subsets of \([a, b]\) such that

\[
\Gamma(f) = \int_{[a, b]} f \, d\mu
\]
for each \( f \in C([a, b]) \). Moreover, \( \| \Gamma \| = \mu([a, b]) \).

**Proof.** First, we will assume that \( \Gamma \) is positive.

For \( a \leq t < b \) and for \( n \) large enough so that \( t + \frac{1}{n} \leq b \), let

\[
\varphi_{t,n}(x) = \begin{cases} 
1 & \text{if } x \in [a, t] \\
1 - n(x - t) & \text{if } x \in (t, t + \frac{1}{n}] \\
0 & \text{if } x \in (t + \frac{1}{n}, b]
\end{cases}
\]

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Note that if \( n \leq m \), then
\[
0 \leq \varphi_{t,m} \leq \varphi_{t,n} \leq 1
\]
It follows that \( \{ \Gamma(\varphi_{t,n}) \} \) is decreasing and bounded below by 0. Therefore, we can define
\[
g(t) = \begin{cases} 
0 & \text{if } t < a \\
\lim_{n \to \infty} \Gamma(\varphi_{t,n}) & \text{if } t \in [a, b) \\
\Gamma(1) & \text{if } t \geq b
\end{cases}
\]
Moreover, if \( t_1 > t \), we have
\[
\varphi_{t,m} \leq \varphi_{t_1,n}.
\]
Since $\Gamma$ is positive, $g(t)$ is monotonically increasing.

It is clear that $g(t)$ is right continuous if $t < a$ or if $t \geq b$. Assume that $t \in [a, b)$. Let $\epsilon > 0$ and choose $n$ large enough so that

$$n > \max(2, \frac{\| \Gamma \|}{\epsilon})$$

and

$$g(t) \leq \Gamma(\varphi_{t,n}) \leq g(t) + \epsilon.$$ 

Let

$$\psi_n(x) = \begin{cases} 
1 & \text{if } x \in [a, t + \frac{1}{n^2}] \\
1 - \frac{n^2}{n^2 - 2} (x - t - \frac{1}{n^2}) & \text{if } x \in (t + \frac{1}{n^2}, t + \frac{1}{n} - \frac{1}{n^2}] \\
0 & \text{if } x \in (t + \frac{1}{n} - \frac{1}{n^2}, b] \end{cases}$$

Then

$$\| \psi_n - \varphi_{t,n} \|_\infty \leq \frac{1}{n}$$

Therefore,

$$\Gamma(\psi_n) \leq \Gamma(\varphi_{t,n}) + \frac{1}{n} \| \Gamma \| \leq g(t) + 2\epsilon.$$
This means that
\[ g(t) \leq g(t + \frac{1}{n^2}) \leq g(t) + 2\epsilon. \]

However, as \( g(t) \) is increasing, this is sufficient to show that \( g(t) \) is right continuous.

The Hahn Extension Theorem gives a Borel measure \( \mu \) such that \( \mu((\alpha, \beta]) = g(\beta) - g(\alpha) \). In particular, if \( a \leq c \leq b \), then
\[ \mu([a, c]) = \mu([a - 1, c]) = g(c). \]

Let \( f \in C([a, b]) \) and let \( \epsilon > 0 \). Let \( \delta \) be such that if \( |x - y| < \delta \) and \( x, y \in [a, b] \), then
\[ |f(x) - f(y)| < \epsilon. \]

Let \( P = \{a = t_0, t_1, \ldots, t_m = b\} \) be a partition with \( \sup(t_k - t_{k-1}) < \frac{\delta}{2} \). Then choose \( n \) large enough so that
\[ \frac{2}{n} < \inf(t_k - t_{k-1}) \]
and
\[ (*) \quad g(t_k) \leq \Gamma(\varphi_{t,n}) \leq g(t_k) + \frac{\epsilon}{m\|f\|_{\infty}}. \]

Next, we let
\[ f_1(x) = f(t_1)\varphi_{t_1,n} + \sum_{k=2}^{m} f(t_k)(\varphi_{t_k,n} - \varphi_{t_{k-1},n}) \]
and
\[ f_2(x) = f(t_1)\chi_{[t_0, t_1]} + \sum_{k=2}^{m} f(t_k)\chi_{[t_{k-1}, t_k]}(x) \]

Note that \( f_1 \) is continuous and piecewise linear. \( f_2 \) is a step function. It is also true that both \( f_1 \) and \( f_2 \) agree with \( f(x) \) at each point \( t_k \) for \( k \geq 1 \). Moreover, the function \( f_1 \) takes on values between \( f(t_{k-1}) \) and \( f(t_k) \) on the interval \([t_{k-1}, t_k]\). As such
\[ \|f_1 - f\|_{\infty} \leq \epsilon \]
and
\[ \sup\{f_2(x) - f(x)\mid x \in [a, b]\} \leq \epsilon. \]

From this we conclude that
\[ |\Gamma(f) - \Gamma(f_1)| \leq \epsilon \|\Gamma\|. \]

We use (*) to see that for \( 2 \leq k \leq m \)
\[ |\Gamma(\varphi_{t_k,n} - \varphi_{t_{k-1},n}) - (g(t_k) - g(t_{k-1}))| \leq \frac{\epsilon}{m\|f\|_{\infty}}. \]

Next, we apply \( \Gamma \) to \( f_1 \) and integrate \( f_2 \) with respect to \( \mu \) to get
\[ |\Gamma(f_1) - \int_{[a, b]} f_2 \, d\mu| \leq \epsilon \]
We also have that
\[ \int_{[a, b]} f_2 \, d\mu = \int_{[a, b]} f \, d\mu \leq \epsilon\mu([a, b]). \]
Therefore,
\[ |\Gamma(f) - \int_{[a, b]} f \, d\mu| \leq \epsilon(2\|\Gamma\| + \mu([a, b])). \]

Since \( \epsilon \) is arbitrary,
\[ \Gamma(f) = \int_{[a, b]} f \, d\mu \]
for each \( f \in C[a, b] \). Moreover, \( \|\Gamma\| = \Gamma(1) = \mu([a, b]). \)

The general result follows from the previous theorem.
6.4 Riesz Representation Theorem for \( C(\Omega) \)

In this section we will briefly discuss how to extend the Riesz Representation to \( C(\Omega) \) when \( (\Omega, d) \) is a compact metric space. In fact we can state this extension in greater generality:

**Theorem 6.4.1.** [Riesz Representation Theorem for \( C(\Omega) \)] Let \( (\Omega, \tau) \) be a compact Hausdorff space. Let \( \Gamma \in C(\Omega)^* \). Then there exists a unique finite regular signed measure \( \mu \) on the Borel subsets of \( \Omega \) such that

\[
\Gamma(f) = \int_{\Omega} f \, d\mu
\]

for each \( f \in C(\Omega) \). Moreover, \( \| \Gamma \| = |\mu| (\Omega) \).

**Remark 6.4.2.** Let \( \mu \in \text{Meas}(\Omega, \mathcal{B}(\Omega)) \). If \( \Gamma_{\mu} \) is defined by

\[
\Gamma_{\mu}(f) = \int_{\Omega} f \, d\mu \quad (\star)
\]

for each \( f \in C(\Omega) \), then \( \Gamma_{\mu} \in C(\Omega)^* \) and

\[
\| \Gamma_{\mu} \| = |\mu| (\Omega) = \|\mu\|_{\text{meas}}.
\]

**Problem 6.4.3.** For the converse how do we construct the measure \( \mu \)?

**Sketch:** We will sketch a solution in the special case where \( (\Omega, d) \) is a compact metric space.

By the Jordan Decomposition Theorem, we may again assume that \( \Gamma \) is positive.

**Key Observation:** Let \( K \subseteq \Omega \) be compact. Assume that \( \{\varphi_n\} \) is a sequence of continuous functions such that

\[
0 \leq \varphi_{n+1}(t) \leq \varphi_n(t) \leq 1
\]

for every \( t \in \Omega \) with

\[
\lim_{n \to \infty} \varphi_n = \chi_K
\]

pointwise. Then

\[
\lim_{n \to \infty} \Gamma(\varphi_n)
\]

exists. Moreover, if \( \mu \) is a measure satisfying (\( \star \)), then the Lebesgue Dominated Convergence Theorem shows that

\[
\mu(K) = \int_{\Omega} \chi_K \, d\mu = \lim_{n \to \infty} \int_{\Omega} \varphi_n \, d\mu = \lim_{n \to \infty} \Gamma(\varphi_n).
\]

From here, let \( K \) be compact. For each \( n \in \mathbb{N} \) let

\[
U_n = \bigcup_{x \in K} B(x, \frac{1}{n})
\]

and let \( F_n = \Omega \setminus U_n \). Then define

\[
\varphi_n(x) = \frac{\text{dist}(x, F_n)}{\text{dist}(x, F_n) + \text{dist}(x, K)}
\]

where \( \text{dist}(x, A) = \inf \{d(x, y) \mid y \in A\} \). Then \( \varphi_n(x) = 1 \) if \( x \in K \) and \( \varphi_n(x) = 0 \) if \( x \in F_n \). Hence \( \varphi_n \to \chi_K \) pointwise.

Moreover since \( \{\text{dist}(x, F_n)\} \) is decreasing, we get

\[
0 \leq \varphi_{n+1}(t) \leq \varphi_n(t) \leq 1.
\]