Chapter 4

Signed Measures

Up until now our measures have always assumed values that were greater than or equal to 0. In this chapter we will extend our definition to allow for both positive and negative values.

4.1 Basic Properties of Signed Measures

DEFINITION **4.1.1.** Let $(X, \mathcal{A})$ be a measurable space. A signed measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to \mathbb{R}^*$ such that

1) μ takes on at most one of the values $-\infty$ or ∞.

2) $\mu(\emptyset) = 0$

3) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is a sequence of pairwise disjoint sets, then

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

EXAMPLE **4.1.2.** 1) Assume that $(X, \mathcal{A})$ is a measurable space and that μ_1, μ_2 are measures on $(X, \mathcal{A})$. If at least one of the μ_i's is finite, then

$$\mu = \mu_1 - \mu_2$$

is a signed measure on $(X, \mathcal{A})$.

2) Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be an integrable function. Then if $f = f^+ - f^-$, we have that

$$\lambda(E) = \int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$$

is a signed measure on $(X, \mathcal{A})$.

4.2 Jordan and Hahn Decompositions

PROBLEM **4.2.1.** *We have just seen that if $(X, \mathcal{A})$ is a measurable space and μ_1, μ_2 are measures on $(X, \mathcal{A})$ with at least one of the μ_i's being finite, then*

$$\mu = \mu_1 - \mu_2$$

is a signed measure on $(X, \mathcal{A})$. We can now ask: do all signed measures arise in this way?

The clue to answering the problem above comes from the second item in our previous example where we decomposed our integral function f into its positive and negative parts. This decomposition, often called the Jordan decomposition of f, can in fact be extended to signed measures, something which we shall see shortly.

We begin with the following natural definitions:

DEFINITION **4.2.2.** Let μ be a signed measure on $(X, \mathcal{A})$. Let $P, N, M \in \mathcal{A}$. Then we say that:

1) P is positive if $\mu(E \cap P) \geq 0$ for all $E \in \mathcal{A}$.

2) N is negative if $\mu(E \cap N) \leq 0$ for all $E \in \mathcal{A}$.

3) M is null if $\mu(E \cap M) = 0$ for all $E \in \mathcal{A}$.

The following useful results follow almost immediately from the definitions above. As such the proofs will be left as exercises.

PROPOSITION **4.2.3.** *Let μ be a signed measure on $(X, \mathcal{A})$.*

1) If $P \in \mathcal{A}$ is positive (negative) [null] and if $E \in \mathcal{A}$ is such that $E \subseteq P$, then E is positive (negative) [null].

2) Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, with each E_n being positive (negative) [null], then $E = \bigcup\limits_{n=1}^{\infty} E_n$ is also positive (negative) [null].

3) Let P be positive and N be negative, then $P \cap N$ is null.

LEMMA **4.2.4.** *Let μ be a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$ be such that $0 < \mu(E) < \infty$. Then there exists a positive set $A \in \mathcal{A}$ with $A \subseteq E$ and $\mu(A) > 0$.*

Proof. If E is positive we are done. So we may assume without loss of generality that E is not positive.

Let $n_1 \in \mathbb{N}$ be the smallest natural number such that there exists an $E_1 \subseteq E$ with

$$\mu(E_1) < -\frac{1}{n_1}.$$

If $E \setminus E_1$ is positive we are done. If not then we choose the smallest natural number $n_2 \in \mathbb{N}$ such that there exists an $E_2 \subseteq E \setminus E_1$ with

$$\mu(E_2) < -\frac{1}{n_2}.$$

Suppose that we have chosen $\{E_1, E_2, \ldots, E_{k-1}\}$ and natural numbers $\{n_1, n_2, \ldots, n_k\}$ as above and that $E \setminus \bigcup\limits_{j=1}^{k-1} E_j$ is not positive, then we choose the smallest natural number $n_k \in \mathbb{N}$ such that there exists an $E_k \subseteq E \setminus \bigcup\limits_{j=1}^{k-1} E_j$ with

$$\mu(E_k) < -\frac{1}{n_k}.$$

Now either this process terminates, in which case we are done, or we are able to inductively construct a seqeunce $\{E_k\}$ as above. In the latter case, let

$$A = E \setminus \bigcup_{k=1}^{\infty} E_k.$$

Since $\{E_k\}$ is pairwise disjoint, and each E_k is disjoint from A, we have that

$$\mu(E) = \mu(A) + \sum_{k=1}^{\infty} \mu(E_k) \qquad (*)$$

From $(*)$ we may conclude the following:

1) Since $\mu(E)$ is finite, the series $\sum\limits_{k=1}^{\infty} \mu(E_k)$ converges.

2) Since the series $\sum\limits_{k=1}^{\infty} \mu(E_k)$ converges, so does the series $\sum\limits_{k=1}^{\infty} \frac{1}{n_k}$, and hence $n_k \to \infty$.

3) Since each $\mu(E_k) < 0$, we have

$$0 < \mu(E) < \mu(A).$$

Assume that there exists an $E_0 \subseteq A$ with $\mu(E_0) < 0$. Then we can find a k large enough so that

$$\mu(E_0) < -\frac{1}{n_k - 1}$$

But we have that

$$E_0 \subseteq A \subseteq E \setminus \bigcup_{j=1}^{k-1} E_j$$

so this would contradict how we went about choosing n_k. It follows that A is positive.

■

THEOREM **4.2.5.** *[Hahn Decomposition Theorem]*
Let μ be a signed measure on $(X, \mathcal{A})$. The there is a positive set $A \in \mathcal{A}$ and a negative set $B \in \mathcal{A}$ so that $X = A \cup B$ and $A \cap B = \emptyset$.

Proof. Since μ cannot take on both $\pm\infty$ we may assume without loss of generality that μ never takes on the value ∞.

Let

$$\alpha = \sup\{\mu(E) | E \in \mathcal{A}, E \text{ is positive}\}.$$

Since $\emptyset$ is positive, we have that $\alpha \geq 0$. We can also find a sequence of positve sets $\{A_i\}$ so that

$$\alpha = \lim_{i \to \infty} \mu(A_i).$$

Next let

$$A = \bigcup_{i=1}^{\infty} A_i$$

Then A is also positive and since $A_i \subseteq A$ for each i we have

$$\mu(A) = \alpha.$$

Let $B = X \setminus A$. Assume that B is not negative. Then there is an $E \subseteq B$ with $\mu(E) > 0$. From the previous lemma we can find a positive set $A_0 \subseteq B$ with $\mu(A_0) > 0$. But then if

$$A^* = A \cup A_0$$

we have that A^* is also positive and since $A \cap A_0 = \emptyset$,

$$\mu(A^*) = \mu(A) + \mu(A_0) > \alpha$$

which is impossible. ■

DEFINITION **4.2.6.** Let μ be a signed measure on $(X, \mathcal{A})$. A pair $\{P, N\}$ of elements in $\mathcal{A}$ for which P is positive N is negative, $P \cup N = X$ and $P \cap N = \emptyset$ is called a Hahn decomposition of X with respect to μ.

REMARK **4.2.7.** It is generally the case that the Hahn decomposition is not unique. In fact, let $X = [0, 1]$ and let $\mathcal{A} = \mathcal{P}(X)$. If $\mu_{\frac{1}{2}}$ is the point mass at $\frac{1}{2}$, then if $P = \{\frac{1}{2}\}$ and $N = [0, 1] \setminus \{\frac{1}{2}\}$, then $\{P, N\}$ is a Hahn decomposition of $[0, 1]$ with respect to μ. However, $P_1 = [0, \frac{1}{2}]$ and $N_1 = (\frac{1}{2}, 1]$ is also a Hahn decomposition.

In fact, if $\{P, N\}$ is a Hahn decomposition of X with respect to μ and if $M \in \mathcal{A}$ is null, then $\{P \cup M, N \setminus M\}$ is a Hahn decomposition of X with respect to μ.

Furthermore, if $\{P_1, N_2\}$ and $\{P_2, N_2\}$ are Hahn decompositions of X with respect to μ, then

$$M = P_1 \Delta P_2 = (P_1 \cap N_2) \cup (N_1 \cap P_2) = N_1 \Delta N_2$$

is a null set.

Furthermore, since $E \cap P_1 \setminus P_2 \subseteq P_1 \Delta P_2$ and $E \cap P_2 \setminus P_1 \subseteq P_1 \Delta P_2$, it follows that

$$\mu(E \cap P_1) = \mu(E \cap P_1 \cap P_2) = \mu(E \cap P_2)$$

for each $E \in \mathcal{A}$.

Similarly,

$$\mu(E \cap N_1) = \mu(E \cap N_1 \cap N_2) = \mu(E \cap N_2)$$

for each $E \in \mathcal{A}$. What is true however, as we shall see, is that every Hahn decomposition induced a decomposition of μ into the difference of two positive measures and that any two Hahn decompositions induce the same decomposition of μ.

DEFINITION **4.2.8.** Let $\{P, N\}$ be a Hahn decomposition of X with respect to μ. Define

$$\mu^+(E) = \mu(E \cap P)$$

and

$$\mu^-(E) = -\mu(E \cap N).$$

REMARK **4.2.9.** It is easy to see that μ^+ and μ^- are both positive measures and that

$$\mu = \mu^+ - \mu^-.$$

The pair $\{\mu^+, \mu^-\}$ is called a Jordan decomposition of μ.

DEFINITION **4.2.10.** Two measures μ and ν on $(X, \mathcal{A})$ are said to be mutually singular if there are disjoint sets $A, B \in \mathcal{A}$ with $X = A \cup B$ and $\mu(A) = 0$ while $\nu(B) = 0$. In this case, we write

$$\mu \perp \nu.$$

THEOREM **4.2.11.** *[Jordan Decomposition Theorem]*

Let μ be a signed measure on $(X, \mathcal{A})$. Then there exist two mutually singular positive measures μ^+ and μ^- such that

$$\mu = \mu^+ - \mu^-.$$

Furthermore, if λ and ν are any two positive measures with

$$\mu = \lambda - \nu,$$

then for each $E \in \mathcal{A}$ we have

$$\lambda(E) \geq \mu^+(E)$$

and

$$\nu(E) \geq \mu^-(E).$$

Finally, if $\lambda \perp \nu$, then $\lambda = \mu^+$ and $\nu = \mu^-$.

Proof. Let $\{\mu^+, \mu^-\}$ be a Jordan decomposition of μ arising from the Hahn decomposition $\{P, N\}$. Then clearly

$$\mu = \mu^+ - \mu^-.$$

Moreover, since

$$\mu^+(N) = \mu(P \cap N) = \mu^-(P) = 0$$

we have that

$$\mu^+ \perp \mu^-.$$

Let λ and ν be any two positive measures with

$$\mu = \lambda - \nu$$

and let $E \in \mathcal{A}$. Then

$$\begin{aligned} \mu^+(E) &= \mu(P \cap E) \\ &= \lambda(P \cap E) - \nu(P \cap E) \\ &\leq \lambda(P \cap E) \\ &\leq \lambda(E) \end{aligned}$$

A similar argument shows that $\nu(E) \geq \mu^-(E)$.

Finally, assume that $\lambda \perp \nu$.

Let $\{A, B\}$ be a partition of X so that $\lambda(B) = 0$ and $\nu(A) = 0$. For each $E \in \mathcal{A}$ we have

$$\mu(E \cap A) = \lambda(E \cap A) - \nu(E \cap A) = \lambda(E \cap A) \geq 0.$$

That is A is positive. Similarly B is negative, so $\{A, B\}$ is a Hahn decomposition. It follows that for an $E \in \mathcal{A}$,

$$\mu^+(E) = \mu(E \cap P) = \mu(E \cap A) = \lambda(E \cap A) = \lambda(E)$$

and

$$\mu^-(E) = -\mu(E \cap N) = -\mu(E \cap B) = \nu(E \cap B) = \nu(E).$$

∎

Definition **4.2.12.** Let μ be a signed measure on $(X, \mathcal{A})$. Then if $\{\mu^+, \mu^-\}$ is the Jordan decomposition of μ, then the measure

$$|\mu| \overset{def}{=} \mu^+ + \mu^-$$

is called the total variation of μ.

Example **4.2.13.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f \in \mathcal{L}(X, \mathcal{A}, \mu)$ be integrable. If

$$\lambda(E) = \int_E f \, d\mu$$

then λ is a signed measure with Jordan decomposition

$$\lambda^+(E) = \int_E f^+ \, d\mu$$

and

$$\lambda^-(E) = \int_E f^- \, d\mu.$$

In particular,

$$|\lambda|(E) = \int_E f^+ \, d\mu + \int_E f^- \, d\mu = \int_E |f| \, d\mu.$$

Definition **4.2.14.** If μ and ν are signed measures on $(X, \mathcal{A})$, then we say that ν is absolutely continuous with respect to μ and write

$$\nu \ll \mu$$

if $|\nu| \ll |\mu|$. We say that μ and ν are mutually singular and write

$$\mu \perp \nu$$

if $|\mu| \perp |\nu|$.

Remark **4.2.15.** 1) If ν is a signed measure and μ is a positive measure on $(X, \mathcal{A})$, then we claim that $\nu \ll \mu$ if and only if whenever $E \in \mathcal{A}$ with $\mu(E) = 0$, we have $\nu(E) = 0$.

Clearly if $\nu \ll \mu$ and $\mu(E) = 0$, we have $\nu(E) = 0$. To prove the converse assume that $\{P, N\}$ is a Hahn-decomposition for ν. Let $E \in \mathcal{A}$ be such that $m(E) = 0$. Then clearly

$$m(E \cap P) = 0 = m(E \cap N).$$

It follows from our assumption that

$$\nu^+(E) = \nu(E \cap P) = 0$$

and

$$\nu^-(E) = \nu(E \cap N) = 0.$$

Therefore, we have that $|\nu|(E) = 0$ and $|\nu| \ll \mu$ as desired.

2) If ν is a signed measure and μ is a positive measure on $(X, \mathcal{A})$, then we claim that $\nu \perp \mu$ if and only if there are disjoint sets $A, B \in \mathcal{A}$ with $X = A \cup B$ and $\mu(A) = 0$ while $\nu(E) = 0$ for any $E \subseteq B, E \in \mathcal{A}$.

If $\nu \perp \mu$, then there are disjoint sets $A, B \in \mathcal{A}$ with $X = A \cup B$ and $\mu(A) = 0$ while $|\nu|(B) = 0$. But then if $E \subseteq B, E \in \mathcal{A}$, we have $|\nu|(E) = 0$ and hence $\nu(E) = 0$.

To see that the converse holds let $A, B \in \mathcal{A}$ be disjoint sets with $X = A \cup B$ and $\mu(A) = 0$ while $\nu(E) = 0$ for any $E \subseteq B, E \in \mathcal{A}$. Let $\{P, N\}$ be a Hahn decomposition for ν. Let $E_1 = B \cap P$ and $E_2 = B \cap n$. Then

$$\mu^+(B) = \mu(E_1) = 0$$

and

$$\mu^-(B) = \mu(E_2) = 0.$$

It follows that $|\nu|(B) = 0$ and that $\nu \perp \mu$.

The following proposition gives us an alternative characterization of absolute continuity for finite positive measures which will prove useful later.

Proposition **4.2.16.** *Let λ and μ be finite positive measures on $(X, \mathcal{A})$. Then the following are equivalent:*

1) $\lambda \ll \mu$

2) For every $\epsilon > 0$ there exist a $\delta > 0$ such that if $E \in \mathcal{A}$ and $\mu(E) < \delta$, then $\lambda(E) < \epsilon$.

Proof. 1) $\Rightarrow$ 2). Suppose that 2 fails. Then we can find an $\epsilon_0 > 0$ and a sequence of sets $\{E_k\} \subseteq \mathcal{A}$ such that $\mu(E_k) < \frac{1}{2^k}$ but $\lambda(E_k) \geq \epsilon_0$.

Now let

$$F_n = \bigcup_{k=n}^{\infty} E_k.$$

It follows that $\mu(F_n) < \frac{1}{2^{n-1}}$ while $\lambda(F_n) \geq \epsilon_0$. From here, since $\{F_n\}$ is decreasing, we get that

$$\mu(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n) = 0,$$

while

$$\lambda(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \lambda(F_n) \geq \epsilon_0.$$

This shows that λ is not absolutely continuous with respect to μ.

2) $\Rightarrow$ 1). Suppose 2) holds and $\mu(E) = 0$. Then for any $\epsilon > 0$ we have that $\mu(E) < \delta$ where $\delta > 0$ is chosen as in 2). It follows that $\lambda(E) < \epsilon$. But since $\epsilon > 0$ was arbitrary, we get that $\lambda(E) = 0$

■

4.3 Radon-Nikodym Theorem

Remark **4.3.1.** We had previously asked about when, given a measure space $(X, \mathcal{A}, \mu)$, and any measure λ on $\mathcal{A}$, does there exists an $f \in \mathcal{M}^+(X, \mathcal{A})$ with the property that for every $E \in \mathcal{A}$,

$$\lambda(E) = \int_E f \, d\mu$$

for all $E \in \mathcal{A}$. We have also seen that if such an f exists then it must be the case that $\lambda \ll \mu$. At this point we are in a position to use what we have learned about decompositions of signed measures to establish the converse of this result in the case of σ-finite measures.

Theorem **4.3.2.** *[Radon-Nikodym Theorem]*

Let λ and μ be σ-finite measures on $(X, \mathcal{A})$. Suppose that λ is absloutely continuous with respect to μ. Then there exists $f \in \mathcal{M}^+(X, \mathcal{A})$ such that

$$\lambda(E) = \int_E f \, d\mu$$

for every $E \in \mathcal{A}$. Moreover f is uniquely determined μ-almost everywhere.

Proof. **Case 1:** We assume that λ, μ are finite. For each $c > 0$ let $\{P(c), N(c)\}$ be an Hahn decomposition for the signed measure $\lambda - c\mu$. Let

$$A_1 = N(c)$$

and for each $k \in \mathbb{N}$ let

$$A_{k+1} = N((k+1)c) \setminus \bigcup_{i=1}^{k} A_i.$$

It follows that $\{A_i\}_{i=1}^{\infty}$ is pairwise disjoint and

$$\bigcup_{i=1}^{k} N(ic) = \bigcup_{i=1}^{k} A_i.$$

Consequently, we have

$$A_k = N(kc) \cap \bigcap_{i=1}^{k-1} P(ic).$$

If $E \in \mathcal{A}$ and $E \subseteq A_k$, then $E \subseteq N(kc)$ and $E \subseteq P((k-1)c)$. As such we have

$$(k-1)c\mu(E) \leq \lambda(E) \leq kc\mu(E). \qquad (*)$$

Next, let
$$B = X \setminus \bigcup_{i=1}^{\infty} A_i = X \setminus \bigcup_{i=1}^{\infty} N(ic) = \bigcap_{i=1}^{\infty} P(ic).$$
Since $B \subseteq P(kc)$ for all $k \in \mathbb{N}$, we get that
$$0 \leq kc\mu(B) \leq \lambda(B) \leq \lambda(X) < \infty$$
for each $k \in \mathbb{N}$. Therefore, $\mu(B) = 0$ and since $\lambda \ll \mu$ we have that $\lambda(B) = 0$, as well.

We now define for each $c > 0$,
$$f_c(x) = \begin{cases} (k-1)c & \text{if } x \in A_k, \\ 0 & \text{if } x \in B. \end{cases}$$
For each $E \in \mathcal{A}$, we have
$$E = (E \cap B) \cup (\bigcup_{i=1}^{\infty} (E \cap A_k)).$$
Applying $(*)$ to each of the component pieces above, we have that
$$\int_E f_c \, d\mu \leq \lambda(E) \leq \int_E (f_c + c) \, d\mu \leq \int_E f_c \, d\mu + c\mu(X).$$

Now for each $n \in \mathbb{N}$ let
$$g_n = f_{\frac{1}{2^n}}.$$
We get
$$\int_E g_n \, d\mu \leq \lambda(E) \leq \int_E g_n \, d\mu + \frac{\mu(X)}{2^n} \qquad (**).$$
If we let $m \geq n$ then $(**)$ tells us that
$$\int_E g_n \, d\mu \leq \lambda(E) \leq \int_E g_m \, d\mu + \frac{\mu(X)}{2^m} \quad \text{and} \quad \int_E g_m \, d\mu \leq \lambda(E) \leq \int_E g_n \, d\mu + \frac{\mu(X)}{2^n}.$$
Combining these two give us that
$$|\int_E g_n \, d\mu - \int_E g_m \, d\mu| \leq \frac{\mu(X)}{2^n}$$
for each $E \in \mathcal{A}$. In particular this holds for $E_1 = \{x \in X | g_n - g_m| \geq 0\}$ and $E_2 = \{x \in X | g_n - g_m| < 0\}$. This allows us to deduce that
$$\int_X |g_n - g_m| \, d\mu \leq \frac{2\mu(X)}{2^n} = \frac{\mu(X)}{2^{n-1}}$$
and hence that $\{g_n\}_{n=1}^{\infty}$ is Cauchy in $L_1(X, \mathcal{A}, \mu)$.

Assume that $g_n \to f$ in $L_1(X, \mathcal{A}, \mu)$. Since $g_n \in \mathcal{M}^+(X, \mathcal{A})$ we can also assume that $f \in \mathcal{M}^+(X, \mathcal{A})$. Moreover, for any $E \in \mathcal{A}$ we have
$$|\int_E g_n \, d\mu - \int_E f \, d\mu| \leq \int_E |g_n - f| \, d\mu \leq \|g_n - f\|_1 \to 0.$$
It then follows from $(**)$ that
$$\lambda(E) = \lim_{n \to \infty} \int_E g_n \, d\mu = \int_E f \, d\mu.$$

Suppose now that $f, h \in \mathcal{M}^+(X, \mathcal{A})$ are such that
$$\int_E f \, d\mu = \lambda(E) = \int_E h \, d\mu$$

for all $E \in \mathcal{A}$. Let $E_1 = \{x \in X | f(x) > h(x)\}$ and $E_2 = \{x \in X | f(x) < h(x)\}$. Since

$$\int_{E_1} f - h\, d\mu = \int_{E_1} f\, d\mu - \int_{E_1} h\, d\mu = \lambda(E_1) - \lambda(E_1) = 0$$

and

$$\int_{E_2} f - h\, d\mu = \int_{E_2} f\, d\mu - \int_{E_2} h\, d\mu = \lambda(E_2) - \lambda(E_2) = 0$$

we have that $\mu(E_1) = \mu(E_2) = 0$ and hence that $f = h$ μ-a.e.

Case 2: Asume that λ and μ are σ-finite. Let $\{X_n\} \subseteq \mathcal{A}$ be an increasing sequence such that $X = \bigcup\limits_{n=1}^{\infty} X_n$, $\lambda(X_n) < \infty$ and $\mu(X_n) < \infty$. Following the first case, we get for each $n \in \mathbb{N}$ a function $f_n \in \mathcal{M}^+(X, \mathcal{A})$ such that $f_{n_{|X_n^c}} \equiv 0$, and if $E \in \mathcal{A}$ with $E \subseteq X_n$, then

$$\lambda(E) = \int_E f_n\, d\mu.$$

If $m \geq n$, then $X_n \subseteq X_m$, and by our previous uniqueness result, $f_n = f_m$ μ-a.e. Let

$$F_n = \sup\{f_1, f_2, \ldots, f_n\}.$$

Then $\{F_n\}$ is an increasing sequence of positive measurable functions with $F_n = f_n$ μ-a.e and $F_n(x) = 0$ for all $x \in X_n^c$. Let

$$f = \lim_{n\to\infty} F_n.$$

If $E \in \mathcal{A}$, then

$$\lambda(E \cap X_n) = \int_E f_n\, d\mu = \int_E F_n\, d\mu.$$

Given that $E \cap X_n \nearrow E$, continuity from below and the Monotone Convergence Theorem shows us that

$$\lambda(E) = \lim_{n\to\infty} \lambda(E \cap X_n) = \lim_{n\to\infty} \int_E F_n\, d\mu = \int_E f\, d\mu.$$

The uniqueness of f is determined as in the finite case.

∎

Remark **4.3.3.** A close look at the proof of the Radon-Nikodym Theorem shows that the construction of the function f resembles differentiation. The next example shows that in fact this is not simply a coincidence.

Example **4.3.4.** Let $F : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $F'(x) > 0$. Then F is strictly increasing. Let μ_F be the Lebesgue-Stieltjes measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ generated by F. Then μ_F is σ-finite and $\mu_F \ll m$. Moreover, the Fundamental Theorem of Calculus shows that

$$\mu_F((a,b]) = F(b) - F(a) = \int_a^b F'(x)\, dx = \int_{[a,b]} F'\, dm = \int_{(a,b]} F'\, dm.$$

From here we can deduce that if $E \in \mathcal{B}(\mathbb{R})$, then

$$\mu_F(E) = \int_E F'\, dm.$$

In particular, the function we would have obtained in via the Radon-Nikodym Theorem is m-a.e equal to F'.

Motivated by the previous example, we have the following definition.

DEFINITION **4.3.5.** The function f whose existence was established in the Radon-Nikodym Theorem is called the Radon-Nikodym derivative of λ with respect to μ and it is denoted by $\frac{d\lambda}{d\mu}$.

REMARK **4.3.6.** 1) $\frac{d\lambda}{d\mu}$ is integrable if and only if λ is a finite measure.

2) In the case that λ is σ- finite, with $X = \bigcup_{n=1}^{\infty} X_n$ and each X_n being such that $\lambda(X_n) < \infty$, we have that

$$\lambda(X_n) = \int_{X_n} \frac{d\lambda}{d\mu}\, d\mu$$

so $\frac{d\lambda}{d\mu}$ must be finite μ-a.e on X_n. As such, we may assume that $\frac{d\lambda}{d\mu}$ is actually finite everywhere on X.

3) Let $(X, \mathcal{A}, \lambda) = (\mathbb{R}, \boldsymbol{M}(\mathbb{R}), m)$ be the ususal Lebesgue measure space. We can define μ on $(\mathbb{R}, \boldsymbol{M}(\mathbb{R}))$ to be the restriction of the counting measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ to $(\mathbb{R}, \boldsymbol{M}(\mathbb{R}))$. Then $m \ll \mu$, since $\mu(E) = 0$ implies $E = \emptyset$. However, there is no $f \in \mathcal{M}^+(\mathbb{R}, \boldsymbol{M}(\mathbb{R}))$ such that

$$m(E) = \int_E f\, d\mu.$$

This shows that the Radon-Nikodym Theorem can fail if μ is not σ-finite. However, we leave it as an exercise to show that the Radon-Nikodym Theorem can be extended to the case where λ is arbitrary.

4) Let μ, λ, ν be σ-finite measures on $(X, \mathcal{A})$ with $\lambda \ll \mu$ and $\nu \ll \mu$.

(i) If $c > 0$, then $c\lambda \ll \mu$ and

$$\frac{d(c\lambda)}{d\mu} = c\frac{d\lambda}{d\mu}.$$

(ii) We have $(\lambda + \nu) \ll \mu$ and

$$\frac{d(\lambda + \nu)}{d\mu} = \frac{d\lambda}{d\mu} + \frac{d\nu}{d\mu}.$$

(iii) If $\nu \ll \lambda$ and $\lambda \ll \mu$, then $\nu \ll \mu$ and

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu}.$$

(iv) If $\mu \ll \lambda$ and $\lambda \ll \mu$, then

$$\frac{d\nu}{d\mu} = \frac{1}{\frac{d\mu}{d\nu}}.$$

Note: All of the equalities above apply almost everywhere. The proofs are left as an exercise.

5) If λ is a signed measure and μ is a σ-finite measure with $\lambda \ll \mu$, then $\lambda^+ \ll \mu$ and $\lambda^- \ll \mu$. In this case, we let

$$\frac{d\lambda}{d\mu} \stackrel{def}{=} \frac{d\lambda^+}{d\mu} - \frac{d\lambda^-}{d\mu}.$$

As an application of the Radon-Nikodym Theorem we present the following fundamental Decomposition Theorem.

THEOREM **4.3.7.** *[Lebesgue Decomposition Theorem]*

Let λ and μ be σ-finite measures on $(X, \mathcal{A})$. Then there exists two measures λ_1 and λ_2 on $(X, \mathcal{A})$ such that $\lambda = \lambda_1 + \lambda_2$, $\lambda_1 \perp \mu$ and $\lambda_2 \ll \mu$. Moreover, these measures are unique.

Proof. Let $\nu = \lambda + \mu$. Then clearly ν is σ-finite, $\lambda \ll \nu$ and $\mu \ll \nu$. It follows that there are functions $f, g \in \mathcal{M}^+(X, \mathcal{A})$ such that

$$\lambda(E) = \int_E f\, d\nu \quad \text{and} \quad \mu(E) = \int_E g\, d\nu.$$

for every $E \in \mathcal{A}$.

Let

$$A = \{x \in X |\ g(x) = 0\}$$

and

$$B = \{x \in X |\ g(x) > 0\}.$$

Then $\{A, B\}$ is a partition of X. Let

$$\lambda_1(E) = \lambda(E \cap A)$$

and

$$\lambda_1(E) = \lambda(E \cap B)$$

for every $E \in \mathcal{A}$. Clearly $\lambda = \lambda_1 + \lambda_2$.

Since

$$\mu(A) = \mu(E) = \int_E g\, d\nu$$

we have that $\lambda_1 \perp \mu$.

If $\mu(E) = 0$, then $\int_E g\, d\nu = 0$ so $g(x) = 0$ for ν-almost every in E. It follows that $\nu(E \cap B) = 0$ and hence that

$$\lambda_2(E) = \lambda(E \cap B) = 0$$

since $\lambda \ll \nu$. That is $\lambda_2 \ll \mu$.

To see that λ_1 and λ_2 are unique we first assume that both λ and μ are finite. Assume also that we can find λ_1 and λ_2 and ν_1 and ν_2 with $\lambda = \lambda_1 + \lambda_2$, $\lambda_1 \perp \mu$ and $\lambda_2 \ll \mu$, $\lambda = \nu_1 + \nu_2$, $\nu_1 \perp \mu$ and $\nu_2 \ll \mu$. Then

$$\gamma = \lambda_1 - \nu_1 = \nu_2 - \lambda_2.$$

In addition γ is such that $\gamma \perp \mu$ and $\gamma \ll \mu$ and hence $\gamma = 0$. (It is an easy exercise to verify this claim).

The case where λ and μ are σ-finite is left as an exercise.

∎