NON-ISOTOPIC LAGRANGIAN TORI IN ELLIPTIC SURFACES

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Abstract. We construct the first example of an infinite family of homotopic but pairwise smoothly non-isotopic Lagrangian tori in the rational elliptic surface $E(1) = \mathbb{P}^2\#9\overline{\mathbb{P}}^2$. Furthermore, we show that any two of these tori are smoothly equivalent, i.e., there exists an orientation-preserving self-diffeomorphism of $E(1)$ that carries one torus to the other. Our construction can be generalized for many other symplectic 4-manifolds, for example, the logarithmic transforms $E(n)_m$ for integers $n, m > 1$.

1. Introduction

Let Σ be a Riemann surface and X a 4-manifold. Two smooth embeddings $f_0, f_1 : \Sigma \to X$ are said to be smoothly *isotopic* if there exists a smooth homotopy $f_t : \Sigma \times [0,1] \to X$ such that for each $t \in [0,1]$, f_t is a smooth embedding of $\Sigma \times \{t\}$. We say that the pairs $(X, f_0(\Sigma))$ and $(X, f_1(\Sigma))$ are smoothly *equivalent* if there exists an orientation-preserving pair-diffeomorphism from one to the other. By isotopy extension theorem, $(X, f_0(\Sigma))$ and $(X, f_1(\Sigma))$ are smoothly equivalent if f_0 and f_1 are smoothly isotopic. In this paper, we will concentrate on the case when the genus of Σ is one, (X, ω) is a symplectic 4-manifold, and the embeddings are Lagrangian, i.e., $f_0^*\omega = f_1^*\omega = 0$. Our main result is the following.

Theorem 1. *Let $m > 1$, $n \geq 1$, $q \neq 0$ be integers. Let $(E(n)_m, \omega)$ be the simply connected elliptic surface with geometric genus $p_g = n - 1$ and a unique multiple fiber F_m of multiplicity m, equipped with a standard Kähler form ω coming from the elliptic fibration. Then there exists an infinite family of homotopic Lagrangian tori representing the same homology class $q[F_m] \in H_2(E(n)_m)$ that are not pairwise smoothly isotopic.*

All homology and cohomology groups in this paper will have integer coefficients. Note that if X is simply connected, then f_0 and f_1 are homotopic if and only if $f_0(\Sigma)$ and $f_1(\Sigma)$ are homologous. Each $E(n)_m$ is indeed simply connected. Also note that the nonexistence of smooth isotopy trivially implies that there does not exist any symplectic isotopy nor Hamiltonian isotopy between our Lagrangian tori. We can also prove the following.

Corollary 2. *Let F be a smooth torus fiber in the rational elliptic surface $E(1) \cong \mathbb{P}^2\#9\overline{\mathbb{P}}^2$. Then there is a symplectic form ω' on $E(1)$ such that, for any nonzero integer q, there exists an infinite family of homotopic and smoothly equivalent Lagrangian tori representing the homology class $q[F] \in H_2(E(1))$ that are not pairwise smoothly isotopic.*

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The symplectic form ω' in Corollary 2 may not necessarily lie in the Kähler cone of $E(1)$. However, since $E(1)$ possesses a unique symplectic structure (see Theorem D in [11]), ω' defines the same symplectic structure as any Kähler form on $E(1)$. We note that all known examples of homotopic but smoothly non-isotopic Lagrangian tori up to now (see [3], [7] and [19]) have also been smoothly inequivalent. We emphasize that the family in Corollary 2 is the first example of Lagrangian tori that are smoothly equivalent but not pairwise smoothly isotopic.

By a standard argument, we can perturb ω' slightly and obtain another symplectic form ω'', with respect to which our tori in Corollary 2, representing $q[F]$ with $q > 0$, are all symplectic submanifolds of $(E(1), \omega'')$. When $q > 1$, these are new examples of homotopic symplectic tori that are smoothly equivalent but not pairwise smoothly isotopic. When $q = 1$, we already know that all symplectic tori representing $[F]$, including the infinite family of non-isotopic symplectic tori constructed in [4], are smoothly equivalent by Lemma 3.3 in [18]. We can also construct such symplectic tori directly in $(E(1), \omega')$ by slightly modifying the construction of Lagrangian tori that follows. By comparing the corresponding Seiberg-Witten invariants, we can easily show that these new symplectic tori are not smoothly equivalent to any of the symplectic tori that were constructed in [4]. We can also obtain a generalization of Theorem 1 by replacing the elliptic surface $E(n)$ with a more general class of symplectic 4-manifolds (see Theorem 14 below).

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2. Generalized link surgery

We present a generalization of the link surgery construction of Fintushel and Stern in [6] as follows. Let $L = \cup_{i=1}^{\ell} K_i \subset S^3$ be an ordered ℓ-component link where K_i is the i-th component. Let $\nu L = \cup_{i=1}^{\ell} \nu K_i$ denote its tubular neighborhood. The complement of νL has boundary $\partial(S^3 \setminus \nu L) = \cup_{i=1}^{\ell} \partial(\nu K_i) \cong \coprod_{i=1}^{\ell} T^2$, a disjoint union of ℓ copies of 2-torus. For each $i = 1, \ldots, \ell$ let X_i be a 4-manifold with a boundary component $T_i^3 \subset \partial X_i$, which is a 3-torus. Let $\varphi_i : T_i^3 \to S^1 \times \partial(\nu K_i)$ be a diffeomorphism between the 3-tori. If T_i^3 and $S^1 \times \partial(\nu K_i)$ are both equipped with boundary orientations, then we require φ_i to be orientation-reversing.

Definition 3. The ordered collection $\mathfrak{D} = \big(\{X_i\}_{i=1}^{\ell}; \{\varphi_i\}_{i=1}^{\ell}\big)$ is called *link surgery gluing data* for an ordered ℓ-component link L. We define the *link surgery manifold corresponding to* $\mathfrak{D}$ to be the 4-manifold (possibly with boundary)

$$L(\mathfrak{D}) \;=\; \Big[\coprod_{i=1}^{\ell} X_i\Big] \bigcup_{\{\varphi_i\}} \big[S^1 \times (S^3 \setminus \nu L)\big]\,,$$

where we identify the T_i^3 boundary component of X_i with the i-th component of the boundary of $S^1 \times (S^3 \setminus \nu L)$ via the gluing diffeomorphism φ_i.

Logarithmic Transformations. For our purposes we only need to look at a particular 2-component link L which is the Hopf link in Figure 1. Let us choose the following oriented factorizations of the boundary:

$$\partial\big[S^1 \times (S^3 \setminus \nu L)\big] \;\cong\; \big[S^1 \times \mu(A) \times \lambda(A)\big] \cup \big[S^1 \times \lambda(B) \times \bar{\mu}(B)\big] \tag{2.1}$$

Here, $\mu(\,\cdot\,)$ and $\lambda(\,\cdot\,)$ denote the meridian and the longitude of a knot, respectively. A bar over μ means negative orientation. Note that both 3-tori are given a boundary orientation coming from an orientation of the 4-manifold $S^1\times(S^3\setminus\nu L)$.

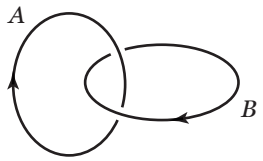

FIGURE 1. Hopf link $L=A\cup B$

Let F denote a generic torus fiber of a simply connected elliptic surface $E(n)$ without any multiple fiber ($n\geq 1$). There is a cartesian product decomposition $F=\rho_1\times\rho_2$, where each $\rho_j\cong S^1$ ($j=1,2$) is an embedded circle in $E(n)$, so that a standard Kähler form (coming from the elliptic fibration) on $E(n)$ restricts to F as $dvol_{\rho_1}\wedge dvol_{\rho_2}$. The elliptic fibration of $E(n)$ also gives a canonical framing $\nu F\cong D^2\times\rho_1\times\rho_2$, and so the complex orientation on $E(n)$ gives rise to the oriented boundary factorization

$$\partial[E(n)\setminus\nu F]\;\cong\;T^3\;\cong\;\bar{\mu}(F)\times\rho_1\times\rho_2\,, \tag{2.2}$$

where $\mu(F)$ denotes the "meridian" of F or the "rim circle" $\partial D^2\times\{\mathrm{pt}\}\subset D^2\times F\cong\nu F$.

We will be working with the following family of link surgery gluing data, indexed by integers $m>1$, which differ only in φ_2:

$$\begin{aligned}\mathfrak{D}_m\;=\;\Big(&\big\{X_1=E(n)\setminus\nu F\,,\,X_2=D^2\times S^1\times S^1\big\}\,;\\ &\Big\{\varphi_1=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix},\;\varphi_2=\begin{bmatrix}-m&1&0\\1&0&0\\0&0&1\end{bmatrix}\Big\}\Big).\end{aligned} \tag{2.3}$$

Here, we have expressed the gluing diffeomorphisms by matrices in $\mathrm{GL}(3,\mathbb{Z})$ representing the induced linear maps $(\varphi_i)_*$ between the first homology groups of the boundary 3-tori with respect to the *ordered* bases corresponding to the oriented factorizations $\partial X_2=\partial D^2\times S^1\times S^1$, (2.1) and (2.2) chosen above.

Lemma 4. *Let L be the Hopf link in S^3. Let $\mathfrak{D}_m$ be link surgery gluing data given by* (2.3)*. Then $L(\mathfrak{D}_m)$ is diffeomorphic to the logarithmic transform $E(n)_m$ of multiplicity m.*

Proof. The proof is easy and therefore omitted. We refer to [4] for an analogous result. □

3. Construction of Lagrangian tori in $E(n)_m$

Our construction of homologically nontrivial Lagrangian tori is a blending of the constructions in [3] and [4]. Let $L=A\cup B$ be the Hopf link. It is easy to see that the Hopf link exterior $S^3\setminus\nu L$ is diffeomorphic to $S^1\times\mathbb{A}$, where $\mathbb{A}\cong S^1\times[\epsilon,1]$ is an annulus and $0<\epsilon<1$ is a constant. We choose the symplectic form

$$\omega_0\;=\;r dr\wedge dx\;+\;dy\wedge d\theta \tag{3.1}$$

on $S^1_x \times (S^3 \setminus \nu L) \cong T^3 \times [\epsilon, 1]$. Here, x is the angular coordinate on the circle S^1_x (the subscript x is used to distinguish this circle from the S^1 factors in $S^3 \setminus \nu L$), y is the angular coordinate parallel to B, and (r, θ) are the polar coordinates on $\mathbb{A}$ (a subset of a normal disk to B) with $\epsilon \leq r \leq 1$.

For each pair of relatively prime integers p and q, draw a closed curve $C_{p,q}$ inside $(S^3 \setminus \nu L)$ such that it is a (p, q) torus knot lying on the torus

$$T(r_0) \,=\, \{(y, r, \theta) \mid r = r_0 = \text{constant},\, \epsilon < r_0 < 1\},$$

which is a radial dilation of $\partial(\nu B) = \{(y, r, \theta) \mid r = \epsilon\}$. The linking numbers are $lk(C_{p,q}, A) = p$ and $lk(C_{p,q}, B) = q$. It follows immediately that $dr|_{C_{p,q}} = 0$. See Figure 2 which illustrates the case when $(p, q) = (1, 4)$.

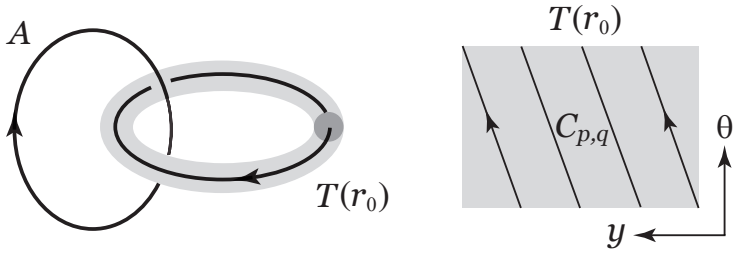

FIGURE 2. Closed curve $C_{p,q}$ on the torus $T(r_0)$

We define a torus $T_{p,q} = S^1_x \times C_{p,q} \subset [S^1_x \times (S^3 \setminus \nu L)] \subset L(\mathfrak{D}_m) = E(n)_m$. We can easily compute that

$$\omega_0|_{T_{p,q}} \,=\, (r dr \wedge dx)|_{S^1_x \times C_{p,q}} + (dy \wedge d\theta)|_{S^1_x \times C_{p,q}} \,=\, 0 + 0 \,=\, 0\,. \tag{3.2}$$

Theorem 5. *For each pair of relatively prime integers p and q, the torus $T_{p,q} = S^1_x \times C_{p,q}$ is a Lagrangian submanifold of $L(\mathfrak{D}_m) \cong E(n)_m$, with respect to a standard Kähler symplectic form on $E(n)_m$, and $[T_{p,q}] = q[F_m] \in H_2(E(n)_m)$, where F_m denotes the multiple fiber.*

Proof. We recall that the standard link surgery gluing data for $E(n)_m$ as in [4] or [6] give the identification of homology classes $[F_m] = [S^1_x \times \mu(B)]$. In Section 2, we have built $E(n)_m$ using "nonstandard" link surgery gluing data $\mathfrak{D}_m$:

$$E(n)_m = [E(n) \setminus \nu F] \cup_{\varphi_1} [S^1_x \times (S^3 \setminus \nu L)] \cup_{\varphi_2} [D^2 \times T^2]\,. \tag{3.3}$$

By carefully comparing the orientations with the standard link surgery gluing data in [4] or [6], we can easily check that $\bar{F}_m$ (F_m with negative orientation) is identified with the core $\{0\} \times T^2$ in the third component of decomposition (3.3).

We now show that $[T_{p,q}] = q[F_m]$ in $H_2(E(n)_m)$. Define $R_1 = \mu(F) \times \rho_1$ and $R_2 = \mu(F) \times \rho_2$, which are homologically essential tori in the boundary of $E(n) \setminus \nu F$. We will write $D^2 \times T^2$ as $D^2 \times S^1_a \times S^1_b$ to distinguish the two circle factors of T^2. From the gluing data (2.3) we identify factorwise:

$$\mu(F) \times \rho_1 \times \rho_2 \;\;=\;\; S^1_x \times \mu(A) \times \lambda(A)\,, \tag{3.4}$$

$$\partial D^2 \times S^1_a \times S^1_b \;\;=\;\; (\lambda(B) - m S^1_x) \times S^1_x \times \bar{\mu}(B). \tag{3.5}$$

Hence $(\varphi_1^{-1})_*[T_{p,q}] = p[R_1] + q[R_2] \in H_2(E(n) \setminus \nu F)$. Note that F gets identified with the boundary torus $\partial(\nu A) = \mu(A) \times \lambda(A)$ in $(S^3 \setminus \nu L)$ by φ_1, and $\bar{F}_m =$

$\{0\}\times S^1_a\times S^1_b$ gets identified with the rim torus $\bar{R}_2=\mu(F)\times\bar{\mu}(B)$ by φ_2. However, note that $[\mu(F)]=0\in H_1(E(n)\setminus\nu F)$. Thus inside $E(n)_m$ we have

$$(\varphi_1)_*[\rho_1]=(\varphi_1)_*([\rho_1]-m[\mu(F)])=[\lambda(B)]-m[S^1_x]=(\varphi_2)_*[\partial D^2]=0\,. \tag{3.6}$$

Hence $[R_1]=[S^1_a\times\partial D^2]=[\partial(S^1_a\times D^2)]=0\in H_2(E(n)_m)$, which implies that $[T_{p,q}]=q[R_2]=q[F_m]$. We also have $0=[(\lambda(B)-mS^1_x)\times\bar{\mu}(B)]=[\lambda(B)\times\bar{\mu}(B)]-m[S^1_x\times\bar{\mu}(B)]$, which implies that $[F]=m[F_m]$.

Next we need to show that the form ω_0 in (3.1) extends to the whole of $E(n)_m$. Recall that F gets identified with the boundary torus $\partial(\nu A)=\mu(A)\times\lambda(A)$ in $(S^3\setminus\nu L)$ by φ_1. It follows that the volume form of F is identified (up to deformation) with the 2-form $dy\wedge d\theta$. Let σ denote a punctured sphere section in $E(n)\setminus\nu F$. Then we see easily that the volume form of σ can be extended to the form $rdr\wedge dx$. We can view (r,x) as a polar coordinate system on the collared boundary of the disk σ. Now we may identify the middle piece in the decomposition (3.3) above as the cylinder

$$[S^1_x\times(S^3\setminus\nu L)]\;\cong\;[S^1_x\times(S^1\times\mathbb{A})]\;\cong\;T^3\times[\epsilon,1].$$

Hence we conclude that ω_0 is, up to deformation, equal to

$$dvol_\sigma\;+\;dvol_F,$$

which is the restriction of the standard Kähler form (coming from the elliptic fibration) of $E(n)\setminus\nu F\cong[E(n)\setminus\nu F]\cup_{\varphi_1}T^3\times[\epsilon,1]$ to the cylinder. Here, *dvol* denotes a volume form.

Since we assume that $m>1$, the logarithmic transform is equivalent to the "rational blowdown" construction of Fintushel and Stern in [5]. On the other hand, by [15] and [16], rational blowdowns can be done symplectically and the symplectic form needs to be modified only near the collar neighborhood of the boundary lens spaces involved. As a result of this and since $F\simeq m\cdot F_m$ remains symplectic in $E(n)_m$, we may assume that the symplectic form on $E(n)_m$ restricts to ω_0 on the subset $T^3\times[\delta,1]\subset[S^1_x\times(S^1\times\mathbb{A})]$ for a suitable constant δ satisfying

$$0<\epsilon<\delta<r_0<1\,.$$

In other words, we may arrange that the sequence of $(m-1)$ blow-ups and the subsequent blowdown of a resulting configuration of spheres occur away from the shorter cylinder $T^3\times[\delta,1]\cong\overline{\nu F}\setminus(T^2\times D_\delta)$, where $\overline{\nu F}\cong T^2\times D^2$ is the closed tubular neighborhood of F and $D_\delta=\{z\in D^2\subset\mathbb{C}\mid|z|<\delta\}$. This, together with (3.2), proves that $T_{p,q}\subset T^3\times\{r_0\}$ is a Lagrangian submanifold of $E(n)_m$. □

Remark 6. Calling the link surgery gluing data $\mathfrak{D}_m$ "nonstandard" is really a misnomer. Note that in a 3-dimensional slice like Figure 2, we are only able to draw two circular dimensions out of the possible three circular dimensions in the cylinder $T^3\times[\epsilon,1]$. Essentially, all we had done differently from the standard identifications in [4] or [6] was to choose to draw the ρ_1 dimension instead of the $\mu(F)$ dimension.

4. Seiberg-Witten and other invariants

Throughout this section m,n,p and q will be integers satisfying: $m>1$, $n\geq1$, $\gcd(p,q)=1$. We start by showing that the fundamental groups of the complements of $T_{p,q}$ are independent of p.

Lemma 7. *We have* $\pi_1(E(n)_m\setminus T_{p,q})\cong\mathbb{Z}/q$.

Proof. We use the well-known fact that $\pi_1(E(n) \setminus \nu F) = 1$. As in [14], it is not too hard to show that

$$\pi_1\big(S^3 \setminus \nu(B \cup C_{p,q})\big) \;\cong\; \frac{\langle \lambda(A) \rangle * \pi_1(\partial(\nu B))}{\lambda(A)^q = \lambda(B)^p \mu(B)^q} \,.$$

Using Van Kampen's Theorem, we compute that

$$\begin{aligned}
& \pi_1(E(n)_m \setminus \nu T_{p,q}) \\
\cong \;& \pi_1\Bigg(E(n) \setminus \nu F \bigcup_{(3.4)} S^1_x \times \big(S^3 \setminus \nu(A \cup B \cup C_{p,q})\big) \bigcup_{(3.5)} D^2 \times S^1_a \times S^1_b\Bigg) \\
\cong \;& \frac{\pi_1(S^3 \setminus \nu(B \cup C_{p,q})) * \pi_1(D^2 \times S^1_a \times S^1_b)}{\{\lambda(A) = 1,\ \lambda(B) = 1,\ S^1_a = 1,\ \mu(B)^{-1} = S^1_b\}} \\
\cong \;& \langle \mu(B) \rangle / \{\mu(B)^q = 1\} \;\cong\; \mathbb{Z}/q \,.
\end{aligned}$$

Note that we have used $S^1_x = \mu(F) = 1$. □

The following lemma says that some of the local symplectic invariants of our tori are independent of both p and q. We refer to [1] and [7] for the definitions.

Lemma 8. *The Lagrangian torus $T_{p,q}$ has Maslov index $0 \in H^1(T_{p,q})$. The Lagrangian framing defect of $C_{p,q}$ is zero.*

Proof. The Maslov index of $T_{p,q}$ is trivial since each circle factor of $T_{p,q} = S^1_x \times C_{p,q}$ has constant slope. It is easy to see that a parallel (p,q) torus knot on $T(r_0)$ is a Lagrangian push-off of $C_{p,q}$, and such a push-off has linking number zero with $C_{p,q}$. □

We shall adhere to the convention that (p,q)-cable means that the cable is p times the longitude and q times the meridian. Let $L_{p,q}$ denote the ordered 3-component link which is the Hopf link $A \cup B$ plus $C_{p,q}$, which is the (p,q)-cable of B. The components of $L_{p,q}$ are ordered lexicographically. Let us choose the following oriented factorizations of the boundary $\partial[S^1 \times (S^3 \setminus \nu L_{p,q})]$

$$\big[S^1 \times \mu(A) \times \lambda(A)\big] \cup \big[S^1 \times \lambda(B) \times \bar{\mu}(B)\big] \cup \big[S^1 \times \mu(C_{p,q}) \times \lambda(C_{p,q})\big] \tag{4.1}$$

Lemma 9. *The Alexander polynomial of $L_{p,q}$ is given by*

$$\Delta_{L_{p,q}}(x,y,z) \;=\; x^p y^q z^{pq} - 1 \,,$$

where where the variables x, y, z refer to the meridians of, respectively, A, B and $C_{p,q}$.

Proof. See [2] for a computation for this and other cabled links. □

Our strategy is to show that for fixed choices of m, n and q, the isotopy types of the tori $\{T_{p,q} \mid \gcd(p,q) = 1\}$, indexed by p, can be distinguished by comparing the Seiberg-Witten invariants of the corresponding family of fiber-sum 4-manifolds $\{E(n)_m \#_{T_{p,q}=F'} E(k) \mid \gcd(p,q) = 1\}$ for some fixed integer $k \geq 1$ and a smooth fiber $F' = \rho'_1 \times \rho'_2 \subset E(k)$. Note that F' has a canonical framing coming from the elliptic fibration structure on $E(k)$. Before we fiber-sum, we use the Lagrangian framing (as defined on p. 952 of [7]) to trivialize the tubular neighborhood of $T_{p,q}$.

Lemma 10. *The fiber sum $E(n)_m\#_{T_{p,q}=F'}E(k)$ is diffeomorphic to the link surgery manifold $L_{p,q}(\mathfrak{D}_{m,k})$, where*

$$\mathfrak{D}_{m,k} \;=\; \Big(\big\{X_1 = E(n)\setminus \nu F\,,\, X_2 = D^2\times S^1\times S^1,\, X_3 = E(k)\setminus \nu F'\big\}\,; \\ \Big\{\varphi_1 = \begin{bmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\; \varphi_2 = \begin{bmatrix} -m & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},\; \varphi_3 = \begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\Big\}\Big).$$

Here, we have chosen the boundary factorization (4.1), *φ_1 and φ_2 are the same maps as in $\mathfrak{D}_m$, and φ_3 identifies factorwise*:

$$\bar\mu(F')\times\rho_1'\times\rho_2' \;=\; \mu(C_{p,q})\times S^1\times\lambda(C_{p,q}).$$

Proof. The proof is easy and thus omitted. □

Theorem 11. *Let $M = S^1\times(S^3\setminus\nu L_{p,q})$, and $\iota : M\to L_{p,q}(\mathfrak{D}_{m,k})$ be the inclusion map. Let $\beta = \iota_*[S^1\times\mu(B)]$, $\gamma = \iota_*[S^1\times\mu(C_{p,q})]\in H_2(L_{p,q}(\mathfrak{D}_{m,k}))$. Then β and γ are both primitive and linearly independent. The Seiberg-Witten invariant of $L_{p,q}(\mathfrak{D}_{m,k})$ is given by*

$$\mathrm{SW}_{L_{p,q}(\mathfrak{D}_{m,k})} \;=\; \frac{\beta^{-q}-\beta^q}{\beta^{-1}-\beta}(\beta^{-m}-\beta^m)^{n-1}(\beta^{-q}\gamma^{pq}-\beta^q\gamma^{-pq})^{k-1}\,. \tag{4.2}$$

Proof. Let $\alpha = \iota_*[S^1\times\mu(A)]\in H_2(L_{p,q}(\mathfrak{D}_{m,k}))$. From the gluing formulas in [13] and [17], we may conclude that

$$\mathrm{SW}_{L_{p,q}(\mathfrak{D}_{m,k})} \;=\; \mathrm{SW}_{E(n)\setminus\nu F}\cdot\mathrm{SW}_{D^2\times T^2}\cdot\mathrm{SW}_{E(k)\setminus\nu F'}\cdot\Delta^{\mathrm{sym}}_{L_{p,q}}(\alpha^2,\beta^2,\gamma^2), \tag{4.3}$$

where $\Delta^{\mathrm{sym}}_{L_{p,q}}(x,y,z) = x^{p/2}y^{q/2}z^{pq/2} - x^{-p/2}y^{-q/2}z^{-pq/2}$ is the symmetrized Alexander polynomial of $L_{p,q}$. Recall from [13] that $\mathrm{SW}_{E(n)\setminus\nu F} = ([F]^{-1}-[F])^{n-1}$, and

$$\mathrm{SW}_{D^2\times T^2} \;=\; \frac{1}{[\{0\}\times T^2]^{-1}-[\{0\}\times T^2]}\,.$$

In the proof of Theorem 5, we have seen that $[\{0\}\times T^2] = [\bar F_m] = [S^1\times\bar\mu(B)]$ and $[F] = m[F_m] = m[S^1\times\mu(B)]$. Thus in (4.3) we may substitute $\mathrm{SW}_{E(n)\setminus\nu F} = (\beta^{-m}-\beta^m)^{n-1}$ and $\mathrm{SW}_{D^2\times T^2} = -1/(\beta^{-1}-\beta)$.

Next note that $[\lambda(B)] = [\mu(A)]+q[\mu(C_{p,q})]\in H_1(M)$. From (3.6), we can easily deduce that $[S^1\times\lambda(B)] = m[S^1_x\times S^1_x] = 0$, which implies that $\alpha = \gamma^{-q}$. Thus in (4.3) we may substitute

$$\Delta^{\mathrm{sym}}_{L_{p,q}}(\alpha^2,\beta^2,\gamma^2) = \alpha^p\beta^q\gamma^{pq} - \alpha^{-p}\beta^{-q}\gamma^{-pq} = \beta^q-\beta^{-q}\,.$$

Since $[\lambda(C_{p,q})] = p[\mu(A)]+q[\mu(B)]\in H_1(M)$, it follows that $[F'] = [S^1\times\lambda(C_{p,q})] = \alpha^p\beta^q = \beta^q\gamma^{-pq}$, and $\mathrm{SW}_{E(k)\setminus\nu F'} = (\beta^{-q}\gamma^{pq}-\beta^q\gamma^{-pq})^{k-1}$. Hence we have shown that (4.3) simplifies to (4.2).

The proof that β and γ are primitive and linearly independent can be carried out as in [4] and [8] and is omitted. □

5. Proofs of Theorem 1 and Corollary 2

In both of the proofs that follow, $m > 1$ will be a fixed integer and its actual value (the multiplicity of the logarithmic transformation) is not important in the proofs.

Proof of Theorem 1. All that remains is to prove that the tori $\{T_{p,q} \mid \gcd(p,q) = 1\}$, representing the homology class $q[F_m]$, are non-isotopic for fixed value of q. We argue as in the proof of Theorem 4.1 in [8]. Let $p \neq p'$ be two integers that are relatively prime to q. Any isotopy of $T_{p,q}$ to $T_{p',q}$ can be extended to a self-diffeomorphism $\Phi : E(n)_m \to E(n)_m$ such that $\Phi(T_{p,q}) = T_{p',q}$. Such isotopy also guarantees that Φ extends to a diffeomorphism

$$\tilde{\Phi} : L_{p,q}(\mathfrak{D}_{m,k}) \longrightarrow L_{p',q}(\mathfrak{D}_{m,k})$$

satisfying $\tilde{\Phi}_*(q\beta) = q\beta'$, where β and β' denote the homology classes of $S^1 \times \mu(B)$ in $L_{p,q}(\mathfrak{D}_{m,k})$ and $L_{p',q}(\mathfrak{D}_{m,k})$ respectively. Since $H_2(L_{p',q}(\mathfrak{D}_{m,k}))$ is torsion-free, we conclude that $\tilde{\Phi}_*(\beta) = \beta'$. Now the 4-manifolds $L_{p,q}(\mathfrak{D}_{m,k})$ and $L_{p',q}(\mathfrak{D}_{m,k})$ have equivalent Seiberg-Witten invariant, i.e.,

$$\tilde{\Phi}_*\big(\mathrm{SW}_{L_{p,q}(\mathfrak{D}_{m,k})}\big) \;=\; \mathrm{SW}_{L_{p',q}(\mathfrak{D}_{m,k})}\,.$$

For simplicity, let us focus on the case when $k = 2$ in (4.2). As was observed on p. 320 of [8], $\tilde{\Phi}_*$ maps the homology class γ to some linear combination of two linearly independent homology classes $\iota_*[S^1 \times \mu(C_{p',q})]$ and $\iota_*[\lambda(C_{p',q}) \times \mu(C_{p',q})]$. Hence the equivalence of Seiberg-Witten invariants implies that $\tilde{\Phi}_*$ maps the homology class $pq\gamma$ to $\pm p'q\gamma'$, where $\gamma' = \iota_*[S^1 \times \mu(C_{p',q})]$. Since γ and γ' are both primitive, this cannot happen for divisibility reasons. □

Proof of Corollary 2. It is well-known (see e.g. [9] or [10]) that for each $m > 1$, there exists a diffeomorphism $\Psi_m : E(1) \to E(1)_m$, sending the homology class $[F]$ to $[F_m]$. Let $\omega' = \Psi_m^* \omega_m$, where ω_m is a standard Kähler form on $E(1)_m$. Then Theorem 1 immediately implies that the family $\{\Psi_m^{-1}(T_{p,q}) \mid \gcd(p,q) = 1\}$ consists of smoothly non-isotopic Lagrangian tori representing $q[F]$.

We claim that $T_{p,q}$ and $T_{p',q}$ are smoothly equivalent in $E(1)_m$ when $p' = p - cq$, and $c \in \mathbb{Z}$. We start by observing that both tori lie on the 3-torus $S^1_x \times T(r_0)$. $T_{p,q}$ and $T_{p',q}$ are smoothly equivalent in $S^1_x \times T(r_0)$, as there exists a self-diffeomorphism ψ of $S^1_x \times T(r_0)$ that carries one to the other. With respect to the ordered basis $\{[S^1_x], [\lambda(B)], [\bar{\mu}(B)]\}$ for $H_1(S^1_x \times T(r_0))$, ψ is determined by the following matrix

$$\psi_* \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \;\in\; \mathrm{SL}(3,\mathbb{Z})\,.$$

Note that $\psi_*[S^1_x] = [S^1_x]$, and $\psi_*[C_{p,q}] = \psi_*(0,p,-q) = (0,p',-q) = [C_{p',q}]$.

By dilating in the r direction (see Section 3), we can easily construct a self-diffeomorphism of $E(n)_m$ that maps $S^1_x \times T(r_0) \cong T^3 \times \{r_0\}$ diffeomorphically onto the boundary 3-torus

$$T^3 \times \{1\} \;\cong\; S^1 \times \mu(A) \times \lambda(A) \;\subset\; \partial[S^1 \times (S^3 \setminus \nu L)] \;\cong\; T^3 \times [\epsilon, 1].$$

Now recall that $T^3 \times \{1\}$ gets identified with the boundary 3-torus $\partial[E(1) \setminus \nu F]$ in the decomposition (3.3). Thus we can view $T_{p,q}$ and $T_{p',q}$ as smoothly equivalent submanifolds of $\partial[E(1) \setminus \nu F]$. But any orientation-preserving self-diffeomorphism of $\partial[E(1) \setminus \nu F]$ extends over $E(1) \setminus \nu F$, according to [12].

It remains to show that ψ can be extended over the other half of $E(1)_m$, namely

$$T^3 \times [\epsilon, r_0] \;\bigcup_{\varphi_2} [D^2 \times S^1 \times S^1]\,,$$

which is diffeomorphic to $D^2 \times S^1 \times S^1$. This is equivalent to showing that the composition $\psi \circ \varphi_2$, given by the matrix product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -m & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -m & 1 & 0 \\ 1 & 0 & c \\ 0 & 0 & 1 \end{bmatrix},$$

extends over $D^2 \times S^1 \times S^1$. This, in turn, is equivalent to exhibiting a self-diffeomorphism ξ of $D^2 \times S^1 \times S^1$ whose restriction $\xi|_\partial$ to the boundary 3-torus satisfies $\psi \circ \varphi_2 = \varphi_2 \circ \xi|_\partial$. Our situation is summarized in the following commutative diagram.

$$\begin{array}{ccccc} S^1_x \times T(r_0) & \xleftarrow[\varphi_2]{} & \partial(D^2 \times S^1 \times S^1) & \hookrightarrow & D^2 \times S^1 \times S^1 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \xi|_\partial} & & \downarrow{\scriptstyle \xi} \\ S^1_x \times T(r_0) & \xleftarrow{\varphi_2} & \partial(D^2 \times S^1 \times S^1) & \hookrightarrow & D^2 \times S^1 \times S^1 \end{array}$$

Now such $\xi|_\partial$ must be given by the matrix

$$\varphi_2^{-1} \circ \psi \circ \varphi_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -m & 1 & 0 \\ 1 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & mc \\ 0 & 0 & 1 \end{bmatrix}$$

on the boundary. Note that, with respect to the ordered basis $\{[\partial D^2], [S^1_a], [S^1_b]\}$ for $H_1(\partial(D^2 \times S^1_a \times S^1_b))$, any self-diffeomorphism of $\partial(D^2 \times S^1_a \times S^1_b)$ given by a matrix in $\mathrm{GL}(3, \mathbb{Z})$ with the first column vector $(\pm 1, 0, 0)$ can be extended over $D^2 \times S^1_a \times S^1_b$ (for example see p. 485 of [9]). It follows that ξ exists.

Thus we conclude that there exists an orientation-preserving self-diffeomorphism Φ' of $E(1)_m$ that carries $T_{p,q}$ diffeomorphically onto $T_{p',q}$. In conclusion, if p_0 is any integer relatively prime to q, then the family $\{\Psi_m^{-1}(T_{p,q}) \mid p \equiv p_0 \mod q\}$ will satisfy the conclusion of Corollary 2. □

Remark 12. Note that unlike the (hypothetical) diffeomorphism Φ in the proof of Theorem 1, the diffeomorphism Φ' in the proof of Corollary 2 cannot extend to a diffeomorphism between the fiber sums $L_{p,q}(\mathfrak{D}_{m,k})$ and $L_{p',q}(\mathfrak{D}_{m,k})$. If there was such an extension $\tilde{\Phi}'$, then just like in the proof of Theorem 1, we could conclude that $\tilde{\Phi}'_*(pq\gamma) = \pm p'q\gamma'$, once again a contradiction. To sum it up, $T_{p,q}$ can be mapped to $T_{p',q}$ by a self-diffeomorphism of $E(1)_m$, but this self-diffeomorphism cannot be an extension of any smooth isotopy between these tori.

6. Generalization to other symplectic 4-manifolds

Theorem 1 can be readily generalized to apply to more general symplectic 4-manifolds. In the following definition, we single out the properties of the pair $(E(n), F)$ that are essential in the proof of Theorem 1.

Definition 13. Let T be a symplectic torus submanifold in a symplectic 4-manifold (X, ω). We say that (X, T, ω) is an *affable* triple if the following four conditions hold.

(i) The tubular neighborhood $(\nu T, \omega|_{\nu T})$ is symplectomorphic to $(D^2 \times S^1 \times S^1, dvol_{D^2} + dvol_{S^1 \times S^1})$.
(ii) T contains a loop, primitive in $\pi_1(T)$, which bounds in $X \setminus T$ an embedded disk of self-intersection -1.

(iii) The Seiberg-Witten invariant $\mathrm{SW}_{X\setminus\nu T}$ is a nonzero polynomial in $[T] \in H_2(X \setminus \nu T)$.
(iv) $H_2(X \setminus \nu F)$ is torsion-free.

The following is a direct generalization of Theorem 1.

Theorem 14. *Suppose $m > 1$, $q \neq 0$ are integers and (X, T, ω) is an affable triple as in Definition* 13. *Let (X_m, ω_m) denote the result of a generalized logarithmic transformation of multiplicity m on T. Let T_m denote the multiple fiber of X_m. Then there exists an infinite family of Lagrangian tori representing the same homology class $q[T_m] \in H_2(X_m)$ that are not pairwise smoothly isotopic.*

Proof. It was shown in Section 8.5 of [10] (see also [5] and [15]) that X_m possesses a canonical symplectic form ω_m coming from the symplectic form ω on X. The rest of the proof is a straightforward modification of the proof of Theorem 1. We need to replace $E(n) \setminus \nu F$ in (2.3) with $X \setminus \nu T$, and replace $\bar{\mu}(F) \times \rho_1 \times \rho_2$ in (2.2) with $\bar{\mu}(T) \times \tau_1 \times \tau_2$, where the factorization $T = \tau_1 \times \tau_2$ comes from the symplectomorphism to $S^1 \times S^1$ in condition (i) of Definition 13. □

References

[1] V. I. Arnol′d: Characteristic class entering in quantization conditions, *Funct. Anal. Appl.* **1** (1967), 1–13.
[2] D. Eisenbud and W. Neumann: *Three-dimensional Link Theory and Invariants of Plane Curve Singularities*, Annals of Mathematics Studies, Vol. 110, Princeton University Press, Princeton, NJ, 1985.
[3] T. Etgü, D. McKinnon and B. D. Park: Lagrangian tori in homotopy elliptic surfaces, *Trans. Amer. Math. Soc.* **357** (2005), 3757–3774.
[4] T. Etgü and B. D. Park: Symplectic tori in rational elliptic surfaces, *Math. Ann.* **334** (2006), 679–691.
[5] R. Fintushel and R. J. Stern: Rational blowdown of smooth 4-manifolds, *J. Differential Geom.* **46** (1997), 181–235.
[6] R. Fintushel and R. J. Stern: Knots, links and 4-manifolds, *Invent. Math.* **134** (1998), 363–400.
[7] R. Fintushel and R. J. Stern: Invariants for Lagrangian tori, *Geom. Topol.* **8** (2004), 947–968.
[8] R. Fintushel and R. J. Stern: Tori in symplectic 4-manifolds, in *Proceedings of the Casson Fest*, Geometry and Topology Monographs, Vol. 7, pp. 311–333, Geometry and Topology Publications, 2004.
[9] R. E. Gompf: Nuclei of elliptic surfaces, *Topology* **30** (1991), 479–511.
[10] R. E. Gompf and A. I. Stipsicz: 4-*Manifolds and Kirby Calculus*, Graduate Studies in Mathematics, Vol. 20, Amer. Math. Soc., 1999.
[11] T. J. Li and A. Liu: Symplectic structure on ruled surfaces and a generalized adjunction formula, *Math. Res. Lett.* **2** (1995), 453–471.
[12] T. Matumoto: Extension problem of diffeomorphisms of a 3-torus over some 4-manifolds, *Hiroshima Math. J.* **14** (1984), 189–201.
[13] B. D. Park: A gluing formula for the Seiberg-Witten invariant along T^3, *Michigan Math. J.* **50** (2002), 593–611.
[14] B. D. Park, M. Poddar and S. Vidussi: Homologous non-isotopic symplectic surfaces of higher genus, *Trans. Amer. Math. Soc.* **359** (2007), 2651–2662.
[15] M. Symington: Symplectic rational blowdowns, *J. Differential Geom.* **50** (1998), 505–518.
[16] M. Symington: Private communication, May 2004.
[17] C. H. Taubes: The Seiberg-Witten invariants and 4-manifolds with essential tori, *Geom. Topol.* **5** (2001), 441–519.
[18] M. Usher: Kodaira dimension and symplectic sums, *Comment. Math. Helv.* **84** (2009), 57–85.
[19] S. Vidussi: Lagrangian surfaces in a fixed homology class: Existence of knotted Lagrangian tori, *J. Differential Geom.* **74** (2006), 507–522.

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