# EXOTIC SMOOTH STRUCTURES ON $S^{2} \times S^{2}$ 

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#### Abstract

We construct an infinite family of mutually nondiffeomorphic irreducible smooth structures on the topological 4-manifold $S^{2} \times S^{2}$.


## 1. Introduction

Let $M$ denote a closed smooth 4-manifold. $M$ is called irreducible if every connected sum decomposition of $M$ as $M=X \# Y$ implies that either $X$ or $Y$ is homeomorphic to the 4 -sphere $S^{4}$. The following lemma will be useful for determining irreducibility.

Lemma 1. Every closed smooth oriented simply connected spin 4-manifold with nontrivial Seiberg-Witten invariant is irreducible.

Proof. Let $M$ be a closed smooth oriented simply connected spin 4-manifold with nontrivial Seiberg-Witten invariant. Suppose $M=X \# Y$ is a connected sum of two smooth 4-manifolds $X$ and $Y$. Then both $X$ and $Y$ are simply connected and the intersection forms of $X$ and $Y$ are both even.

If $b_{2}^{+}(X)$ and $b_{2}^{+}(Y)$ are both strictly positive, then the Seiberg-Witten invariant of $X \# Y$ is trivial (cf. [22]). This contradiction shows that one of $b_{2}^{+}(X)$ and $b_{2}^{+}(Y)$ is 0 . Without loss of generality, assume $b_{2}^{+}(X)=0$. If $b_{2}(X)=b_{2}^{-}(X)>0$, then the intersection form of $X$ is a nontrivial negative definite form, so by Donaldson's theorem in [7], it is isomorphic to the diagonal form $b_{2}(X)[-1]$. But this contradicts the fact that the intersection form of $X$ is even. Thus we conclude that $b_{2}(X)=$ 0 . Since $X$ is simply connected, $X$ must be homeomorphic to $S^{4}$ by Freedman's theorem in [9].

To state our results, it will be convenient to introduce the following terminology.
Definition 2. Let $M$ be a smooth 4-manifold. We say that $M$ has $\infty$-property if there exist an irreducible symplectic 4 -manifold and infinitely many mutually nondiffeomorphic irreducible nonsymplectic 4-manifolds, all of which are homeomorphic to $M$.

Let $S^{2} \times S^{2}$ denote the cartesian product of two 2 -spheres. It was proved in [3] that $(2 k-1)\left(S^{2} \times S^{2}\right)$, the connected sum of $2 k-1$ copies of $S^{2} \times S^{2}$, has $\infty$-property for every integer $k \geq 138$. The main goal of this paper is to prove the following.

Theorem 3. $S^{2} \times S^{2}$ has $\infty$-property.

[^0]At the moment, $S^{2} \times S^{2}$ has the smallest Euler characteristic (four) amongst all closed simply connected topological 4-manifolds that are known to possess more than one smooth structure. The only closed simply connected 4 -manifolds with smaller Euler characteristic are $S^{4}$ and the complex projective plane $\mathbb{C P}^{2}$. In [2], we show that $(2 k-1)\left(S^{2} \times S^{2}\right)$ has $\infty$-property for every $k \geq 2$. From Theorem 3 , we also obtain infinitely many mutually nondiffeomorphic (albeit not irreducible) smooth structures on $\mathbb{C P}^{2} \# m \overline{\mathbb{C P}}^{2}$ for every $m \geq 2$ by blowing up $m-1$ points on our exotic $S^{2} \times S^{2}$, s . When $m=4$, we can deduce the following.

Corollary 4. There exist infinitely many mutually nondiffeomorphic 4-manifolds $\left\{Y_{n} \mid n=1,2,3, \ldots\right\}$ that are all homeomorphic to $\mathbb{C P}^{2} \# 4 \overline{\mathbb{C P}}^{2}$ and satisfy the following.
(i) Each $Y_{n}$ does not admit any Einstein metric.
(ii) Each $Y_{n}$ has negative Yamabe invariant.
(iii) On each $Y_{n}$, there does not exist any nonsingular solution to the normalized Ricci flow for any initial metric.

Proof. In Section 3, we construct infinitely many mutually nondiffeomorphic 4manifolds $\left\{M_{n}^{1} \mid n=1,2,3, \ldots\right\}$ that are all homeomorphic to $S^{2} \times S^{2}$ and have distinct nontrivial Seiberg-Witten invariants. Let $Y_{n}=M_{n}^{1} \# 3 \overline{\mathbb{C P}}^{2}$. Then by Freedman's theorem in [9], each $Y_{n}$ is homeomorphic to $\mathbb{C P}^{2} \# 4 \overline{\mathbb{C P}}^{2}$.

Part (i) now follows from Theorem 3.3 in [15] (by setting $X=M_{n}^{1}, k=3$ and $\ell=0$ in the notation of [15]). Part (ii) follows from [14] (see the paragraph preceding Theorem 7 in [14]). Part (iii) follows from Theorem A in [13] (by setting $X=M_{n}^{1}$ and $k=3$ in the notation of [13]).

We point out that the analogue of Corollary 4 for $\mathbb{C P}^{2} \# m \overline{\mathbb{C P}}^{2}$, when $m=$ $5,6,7,8$, has been proved in $[13,18]$. The proof of Theorem 3 is spread out in Sections 2-5. In Section 4, we also construct other families of 4-manifolds with cyclic fundamental groups and having Euler characteristic equal to four. Our overall strategy is to apply the 'reverse engineering' technique of [8] to a suitably chosen nontrivial genus 2 surface bundle over a genus 2 surface. In [5, 6], Baykur has used a similar method to construct exotic smooth structures on $\mathbb{C P}^{2} \# 5 \overline{\mathbb{C P}}^{2}$.

## 2. Model complex surface

Let $\Sigma_{g}$ denote a closed genus $g$ Riemann surface. Let $\tau_{1}: \Sigma_{2} \rightarrow \Sigma_{2}$ be an elliptic involution with two fixed points $\left\{z_{0}, z_{1}\right\}$ such that $\Sigma_{2} /\left\langle\tau_{1}\right\rangle=\Sigma_{1}$. Let $\tau_{2}: \Sigma_{3} \rightarrow \Sigma_{3}$ be a fixed point free involution with $\Sigma_{3} /\left\langle\tau_{2}\right\rangle=\Sigma_{2}$. See Figure 1 and also Figure 2 wherein $\tau_{1}$ is a 180 degree anti-clockwise rotation around the 'center' point $z_{1}$. There is a free $\mathbb{Z} / 2$ action on the product $\Sigma_{2} \times \Sigma_{3}$ given by $\alpha(z, w)=\left(\tau_{1}(z), \tau_{2}(w)\right)$ for $\alpha \neq 0 \in \mathbb{Z} / 2, z \in \Sigma_{2}$, and $w \in \Sigma_{3}$. Let $X=\left(\Sigma_{2} \times \Sigma_{3}\right) /\langle\alpha\rangle$ denote the quotient manifold, and let $q: \Sigma_{2} \times \Sigma_{3} \rightarrow X$ denote the quotient map. The Euler characteristic and the Betti numbers of $X$ are $e(X)=e\left(\Sigma_{2} \times \Sigma_{3}\right) / 2=(-2)(-4) / 2=$ $4, b_{1}(X)=6$ and $b_{2}(X)=14$. $X$ is a minimal complex surface of general type with $p_{g}=q=3$ and $K^{2}=8$ (cf. [11]).

Let $\left\{a_{1}, b_{1}, a_{2}, b_{2}\right\}$ and $\left\{c_{1}, d_{1}, c_{2}, d_{2}, c_{3}, d_{3}\right\}$ be the set of simple closed curves representing the standard generators of $\pi_{1}\left(\Sigma_{2}, z_{0}\right)$ and $\pi_{1}\left(\Sigma_{3}, w_{0}\right)$, respectively. Note that the base point $z_{0}$ of $\Sigma_{2}$ is one of the two fixed points of $\tau_{1}$. Let $w_{0}$ be the base point of $\Sigma_{3}$ and let $w_{1}=\tau_{2}\left(w_{0}\right)$ be as drawn in Figure 1. Throughout,


Figure 1. Involution $\alpha=\left(\tau_{1}, \tau_{2}\right)$


Figure 2. Lifts of Lagrangian tori
we choose $\left\{z_{0}\right\} \times\left\{w_{0}\right\}$ and $q\left(\left\{z_{0}\right\} \times\left\{w_{0}\right\}\right)$ as the base points of $\pi_{1}\left(\Sigma_{2} \times \Sigma_{3}\right)$ and $\pi_{1}(X)$, respectively. After isotopy and by changing the orientations of the curves $a_{2}$ and $b_{2}$ if necessary, we can assume that $\alpha_{*}\left(a_{1} \times\left\{w_{0}\right\}\right)=a_{2} \times\left\{w_{1}\right\}$ and $\alpha_{*}\left(b_{1} \times\left\{w_{0}\right\}\right)=b_{2} \times\left\{w_{1}\right\}$.

Let $\tilde{c}_{2}$ be a path from $w_{0}$ to $w_{1}$ in $\Sigma_{3}$ shown in Figures 1 and 2. Since $\tau_{2}$ maps the endpoints of $\tilde{c}_{2}$ to one another, $q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)$ is a closed path in $X$. Note that $\tau_{2}\left(\tilde{c}_{2}\right)$ is a path from $w_{1}$ to $w_{0}$, and satisfy $q\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}\right)\right)=q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)$ since $z_{0}$ is a fixed point of $\tau_{1}$. It follows that $q_{*}\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)^{2}=q_{*}\left(\left\{z_{0}\right\} \times c_{2}\right)$ in $\pi_{1}(X)$. Similarly, $\tau_{2}\left(\tilde{c}_{2}^{-1}\right)$ is a path from $w_{0}$ to $w_{1}$ that traverse along the 'bottom' half of the loop $c_{2}$ in the clockwise direction in Figure 1.

Note that the loops $a_{2} \times\left\{w_{0}\right\}$ and $a_{2} \times\left\{w_{1}\right\}$ are freely homotopic along the path $\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}^{-1}\right)$ in $\Sigma_{2} \times \Sigma_{3}$. Thus we have

$$
\begin{align*}
q_{*}\left(a_{2} \times\left\{w_{0}\right\}\right) & =q_{*}\left(\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}^{-1}\right)\right) \cdot\left(a_{2} \times\left\{w_{1}\right\}\right) \cdot\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}\right)\right)\right) \\
& =q_{*}\left(\left(\left\{z_{0}\right\} \times \tilde{c}_{2}^{-1}\right) \cdot\left(a_{1} \times\left\{w_{0}\right\}\right) \cdot\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)\right) \tag{1}
\end{align*}
$$

inside $\pi_{1}(X)$. In this paper, the order of path compositions will always be from left to right. An explicit homotopy $F_{1}:[0,1] \times[0,1] \rightarrow X$ is given by

$$
F_{1}(s, t)=\left\{\begin{array}{lll}
q\left(a_{2}(0) \times \tau_{2}\left(\tilde{c}_{2}(1-3 t)\right)\right) & \text { if } & 0 \leq t \leq s / 3  \tag{2}\\
q\left(a_{2}\left(\frac{3 t-s}{3-2 s}\right) \times \tau_{2}\left(\tilde{c}_{2}(1-s)\right)\right) & \text { if } & s / 3 \leq t \leq(3-s) / 3 \\
q\left(a_{2}(1) \times \tau_{2}\left(\tilde{c}_{2}(3 t-2)\right)\right) & \text { if } & (3-s) / 3 \leq t \leq 1
\end{array}\right.
$$

where $a_{2}(t)$ and $\tilde{c}_{2}(t)$ are parameterizations of the curves $a_{2}$ and $\tilde{c}_{2}$ satisfying $a_{2}(0)=a_{2}(1)=z_{0}, \tilde{c}_{2}(0)=w_{0}$, and $\tilde{c}_{2}(1)=w_{1}$. Similarly, we have

$$
\begin{equation*}
q_{*}\left(b_{2} \times\left\{w_{0}\right\}\right)=q_{*}\left(\left(\left\{z_{0}\right\} \times \tilde{c}_{2}^{-1}\right) \cdot\left(b_{1} \times\left\{w_{0}\right\}\right) \cdot\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)\right) \tag{3}
\end{equation*}
$$

inside $\pi_{1}(X)$ via the based homotopy

$$
F_{2}(s, t)=\left\{\begin{array}{lll}
q\left(b_{2}(0) \times \tau_{2}\left(\tilde{c}_{2}(1-3 t)\right)\right) & \text { if } \quad 0 \leq t \leq s / 3  \tag{4}\\
q\left(b_{2}\left(\frac{3 t-s}{3-2 s}\right) \times \tau_{2}\left(\tilde{c}_{2}(1-s)\right)\right) & \text { if } \quad s / 3 \leq t \leq(3-s) / 3 \\
q\left(b_{2}(1) \times \tau_{2}\left(\tilde{c}_{2}(3 t-2)\right)\right) & \text { if } \quad(3-s) / 3 \leq t \leq 1
\end{array}\right.
$$

By using based homotopies supported inside $q\left(\left\{z_{0}\right\} \times \Sigma_{3}\right)$, we also deduce that

$$
\begin{aligned}
q_{*}\left(\left\{z_{0}\right\} \times c_{3}\right) & =q_{*}\left(\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}^{-1}\right)\right) \cdot\left(\left\{z_{0}\right\} \times \tau_{2}\left(c_{1}\right)\right) \cdot\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}\right)\right)\right) \\
& =q_{*}\left(\left(\left\{z_{0}\right\} \times \tilde{c}_{2}^{-1}\right) \cdot\left(\left\{z_{0}\right\} \times c_{1}\right) \cdot\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)\right) \\
q_{*}\left(\left\{z_{0}\right\} \times d_{3}\right) & =q_{*}\left(\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}^{-1}\right)\right) \cdot\left(\left\{z_{0}\right\} \times \tau_{2}\left(d_{1}\right)\right) \cdot\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}\right)\right)\right) \\
& =q_{*}\left(\left(\left\{z_{0}\right\} \times \tilde{c}_{2}^{-1}\right) \cdot\left(\left\{z_{0}\right\} \times d_{1}\right) \cdot\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)\right)
\end{aligned}
$$

Note again that, to go from $w_{0}$ to $w_{1}$, we have used the 'bottom' half of the loop $c_{2}$ and traversed it in the clockwise direction in Figure 1. This is why we have $q\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}^{-1}\right)\right)=q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}^{-1}\right)$ first.

The quotient group $\pi_{1}(X) / q_{*}\left(\pi_{1}\left(\Sigma_{2} \times \Sigma_{3}\right)\right)$ is isomorphic to $\mathbb{Z} / 2$, and is generated by the coset of $q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)$. In summary, we have proved the following.

Lemma 5. The following six loops generate $\pi_{1}(X)$ :

$$
\begin{array}{ccc}
q\left(a_{1} \times\left\{w_{0}\right\}\right), & q\left(b_{1} \times\left\{w_{0}\right\}\right), & q\left(\left\{z_{0}\right\} \times c_{1}\right), \\
q\left(\left\{z_{0}\right\} \times d_{1}\right), & q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right), & q\left(\left\{z_{0}\right\} \times d_{2}\right)
\end{array}
$$

From now on, we will sometimes abuse notation and write $a_{1}=q\left(a_{1} \times\left\{w_{0}\right\}\right)$ and $c_{1}=q\left(\left\{z_{0}\right\} \times c_{1}\right)$, etc. The intersection form of $X$ is isomorphic to $7 H$, where

$$
H=\left[\begin{array}{ll}
0 & 1  \tag{5}\\
1 & 0
\end{array}\right]
$$

Hence $\sigma(X)$, the signature of $X$, is 0 . A basis for the intersection form of $X$ is given by the following seven geometrically dual pairs:

$$
\begin{array}{cl}
\left(\left[a_{1} \times c_{1}\right],-\left[b_{1} \times d_{1}\right]\right), & \left(\left[a_{1} \times d_{1}\right],\left[b_{1} \times c_{1}\right]\right), \\
\left(\left[a_{2} \times c_{1}\right],-\left[b_{2} \times d_{1}\right]\right), & \left(\left[a_{2} \times d_{1}\right],\left[b_{2} \times c_{1}\right]\right), \\
\left(\left[\left(\tilde{a}_{1} \tilde{a}_{2}\right) \times \tilde{c}_{2}\right],-\left[b_{1} \times d_{2}\right]\right), & \left(\left[a_{1} \times d_{2}\right],\left[\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}\right]\right), \\
\left(\left[\Sigma_{2} \times\left\{w_{0}\right\}\right],\left[\left\{z_{0}\right\} \times \Sigma_{3}\right]\right) .
\end{array}
$$

Here, [.] denotes the homology class of the image $q(\cdot)$ in the quotient manifold $X$ for short. Note that even though $\left[a_{1} \times\left\{w_{0}\right\}\right]=\left[a_{2} \times\left\{w_{0}\right\}\right]$ in $H_{1}(X ; \mathbb{Z})$, we still have $\left[a_{1} \times c_{1}\right] \neq\left[a_{2} \times c_{1}\right]$ since $\alpha_{*}\left[a_{1} \times c_{1}\right]=\left[a_{2} \times c_{3}\right]$ in $H_{2}\left(\Sigma_{2} \times \Sigma_{3} ; \mathbb{Z}\right)$. The minus signs are there to ensure that the nonzero intersection numbers are +1 with respect to standard orientations. For example,

$$
\begin{aligned}
{\left[a_{1} \times c_{1}\right] \cdot\left[b_{1} \times d_{1}\right] } & =(-1)^{\operatorname{deg}\left(c_{1}\right) \operatorname{deg}\left(b_{1}\right)}\left(a_{1} \cdot b_{1}\right)\left(c_{1} \cdot d_{1}\right) \\
& =-1 \cdot 1 \cdot 1=-1 \\
{\left[a_{1} \times d_{1}\right] \cdot\left[b_{1} \times c_{1}\right] } & =(-1)^{\operatorname{deg}\left(d_{1}\right) \operatorname{deg}\left(b_{1}\right)}\left(a_{1} \cdot b_{1}\right)\left(d_{1} \cdot c_{1}\right) \\
& =(-1) \cdot 1 \cdot(-1)=1 .
\end{aligned}
$$

The composite loop $\tilde{b}_{1} \tilde{b}_{2}$ starts at a point $y_{0}$ on $a_{2}$ and traverses first along $\tilde{b}_{1}$ to the point $y_{1}=\tau_{1}\left(y_{0}\right)$ on $a_{1}$ and then along $\tilde{b}_{2}$ back to the starting point $y_{0}$. See Figure 2. We define the loop $\tilde{a}_{1} \tilde{a}_{2}$ in a similar manner. Both loops $\tilde{a}_{1} \tilde{a}_{2}$ and $\tilde{b}_{1} \tilde{b}_{2}$ are mapped to themselves under $\tau_{1}$. Since the endpoints of $\tilde{c}_{2}$ are mapped to each other under $\tau_{2}$, the cylinders $\left(\tilde{a}_{1} \tilde{a}_{2}\right) \times \tilde{c}_{2}$ and $\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}$ become closed tori in $X$.

As observed in Example 2 of [11], it is convenient to view $X$ as the total space of a genus 2 surface bundle over a genus 2 surface:

$$
\begin{equation*}
X=\frac{\Sigma_{2} \times \Sigma_{3}}{\langle\alpha\rangle} \xrightarrow{f} \frac{\Sigma_{3}}{\left\langle\tau_{2}\right\rangle}=\Sigma_{2} . \tag{6}
\end{equation*}
$$

The quotients $q\left(\Sigma_{2} \times\left\{w_{0}\right\}\right)$ and $q\left(\left\{z_{0}\right\} \times \Sigma_{3}\right)$ are genus 2 surfaces in $X$ that form a fiber and a section of this bundle. Thus $\left(\left[\Sigma_{2} \times\left\{w_{0}\right\}\right],\left[\left\{z_{0}\right\} \times \Sigma_{3}\right]\right)$ is represented by a pair of transversely intersecting genus 2 symplectic surfaces in $X$.

Lemma 6. The loops $q\left(\left\{z_{0}\right\} \times c_{1}\right), q\left(\left\{z_{0}\right\} \times d_{1}\right), q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)$ and $q\left(\left\{z_{0}\right\} \times d_{2}\right)$ represent elements of infinite order in $\pi_{1}(X)$.

Proof. From the homotopy long exact sequence for a fibration, we obtain

$$
0 \longrightarrow \pi_{1}\left(\Sigma_{2}\right) \longrightarrow \pi_{1}(X) \xrightarrow{f_{*}} \pi_{1}\left(\Sigma_{2}\right) \longrightarrow 0
$$

where $f_{*}$ is the homomorphism induced by the bundle map in (6). $f_{*}$ maps the loops in our lemma to the standard generators of $\pi_{1}\left(\Sigma_{2}\right)$ and hence these loops cannot be torsion elements of $\pi_{1}(X)$.

## 3. Construction of exotic $S^{2} \times S^{2}$

Choose $\tau_{1}$ and $\tau_{2}$ invariant volume forms on $\Sigma_{2}$ and $\Sigma_{3}$, respectively. By pushing forward the sum of pullbacks of these volume forms on $\Sigma_{2} \times \Sigma_{3}$ under $q$, we obtain a symplectic form $\omega$ on $X$. Alternatively, we can equip $X$ with a symplectic form
coming from the bundle structure in (6). Now consider the following six Lagrangian tori in $X$ :

$$
\begin{array}{ccc}
q\left(a_{1}^{\prime} \times c_{1}^{\prime}\right), & q\left(b_{1}^{\prime} \times c_{1}^{\prime \prime}\right), & q\left(a_{2}^{\prime} \times c_{1}^{\prime}\right),  \tag{7}\\
q\left(a_{2}^{\prime \prime} \times d_{1}^{\prime}\right), & q\left(b_{1}^{\prime} \times d_{2}^{\prime}\right), & q\left(\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}\right)
\end{array}
$$

The lifts of these Lagrangian tori in $\Sigma_{2} \times \Sigma_{3}$ are drawn in Figure 2. The prime and double prime notations are explained in [8]. Each tori in (7) is Lagrangian with respect to $\omega$ since the first circle factor lies in the $\Sigma_{2}$ direction whereas the second circle factor lies in the $\Sigma_{3}$ direction.

The $\tilde{c}_{2}^{\prime}$ path that is 'parallel' to $\tau_{2}\left(\tilde{c}_{2}\right)$ is drawn in Figures 2 and 3. The endpoints of $\tilde{c}_{2}^{\prime}, v_{0}$ and $v_{1}$, are mapped to each other by $\tau_{2}$. Thus the cylinder $\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}$ in $\Sigma_{2} \times \Sigma_{3}$ becomes a closed torus in $X$. Let $\tilde{b}_{1}(t), \tilde{b}_{2}(t)$ and $\tilde{c}_{2}^{\prime}(t)$ be parameterizations of the curves $\tilde{b}_{1}, \tilde{b}_{2}$ and $\tilde{c}_{2}^{\prime}$, respectively, satisfying $\tilde{b}_{1}(0)=y_{0}, \tilde{b}_{1}(1)=y_{1}, \tilde{b}_{2}(0)=y_{1}$, $\tilde{b}_{2}(1)=y_{0}, \tilde{c}_{2}^{\prime}(0)=v_{1}$ and $\tilde{c}_{2}^{\prime}(1)=v_{0}$. Note that $q\left(\{z\} \times \tilde{c}_{2}^{\prime}\right)$ is not a closed loop for any point $z \in \tilde{b}_{1} \tilde{b}_{2}$ since there is no fixed point of $\tau_{1}$ on $\tilde{b}_{1} \tilde{b}_{2}$. However, the following composition of paths gives rise to a simple closed curve on the $q\left(\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}\right)$ torus:

$$
\beta(t)=\left\{\begin{array}{lll}
\left\{y_{0}\right\} \times \tilde{c}_{2}^{\prime}(2 t) & \text { if } & 0 \leq t \leq 1 / 2  \tag{8}\\
\tilde{b}_{1}(2 t-1) \times\left\{v_{0}\right\} & \text { if } & 1 / 2 \leq t \leq 1
\end{array}\right.
$$

Since $q\left(\left\{y_{0}\right\} \times\left\{v_{1}\right\}\right)=q\left(\left\{y_{1}\right\} \times\left\{v_{0}\right\}\right)$ when $t=0,1$, the loop $q(\beta(t))$ is well defined.
Let $\eta$ be a short 'diagonal' path on the $a_{2} \times \tau_{2}\left(d_{2}\right)$ torus from $\left\{z_{0}\right\} \times\left\{w_{1}\right\}$ to $\left\{y_{0}\right\} \times\left\{v_{1}\right\}$. Then $\alpha(\eta)$ is a path on the $a_{1} \times d_{2}$ torus from $\left\{z_{0}\right\} \times\left\{w_{0}\right\}$ to $\left\{y_{1}\right\} \times\left\{v_{0}\right\}$. Since $q\left(\left\{z_{0}\right\} \times\left\{w_{1}\right\}\right)=q\left(\left\{z_{0}\right\} \times\left\{w_{0}\right\}\right)$, the image $q\left(\eta \cdot \beta \cdot \alpha(\eta)^{-1}\right)$ represents $\tilde{c}_{2} b_{1}$ in $\pi_{1}(X)$. Next consider the composition

$$
\xi(t)=\left\{\begin{array}{lll}
\left\{y_{0}\right\} \times \tilde{c}_{2}^{\prime}(4 t) & \text { if } & 0 \leq t \leq 1 / 4  \tag{9}\\
\tilde{b}_{1}(4 t-1) \times\left\{v_{0}\right\} & \text { if } & 1 / 4 \leq t \leq 1 / 2 \\
\tilde{b}_{2}(4 t-2) \times\left\{v_{0}\right\} & \text { if } & 1 / 2 \leq t \leq 3 / 4 \\
\left\{y_{0}\right\} \times \tilde{c}_{2}^{\prime}(4-4 t) & \text { if } & 3 / 4 \leq t \leq 1
\end{array}\right.
$$

Note that $q\left(\eta \cdot \xi \cdot \eta^{-1}\right)$ represents $\tilde{c}_{2} b_{1} b_{2} \tilde{c}_{2}^{-1}=b_{2} b_{1}$ in $\pi_{1}(X)$.
Both paths $\beta$ and $\xi$ begin at the point $\left\{y_{0}\right\} \times\left\{v_{1}\right\}$, and their images under $q$ represent standard generators for the image of the fundamental group of the $q\left(\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}\right)$ torus in $X$ that are based at $q\left(\left\{y_{0}\right\} \times\left\{v_{1}\right\}\right)$. In particular, the words for the $q\left(\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}\right)$ torus in $\pi_{1}(X)$ are given by the conjugates of

$$
\left(\tilde{c}_{2} b_{1} b_{2} \tilde{c}_{2}^{-1}\right)\left(\tilde{c}_{2} b_{1}\right)\left(\tilde{c}_{2} b_{1} b_{2} \tilde{c}_{2}^{-1}\right)^{-1}\left(\tilde{c}_{2} b_{1}\right)^{-1}=\tilde{c}_{2} b_{1} b_{2} \tilde{c}_{2}^{-1} b_{1}^{-1} b_{2}^{-1}
$$



Figure 3. $\tilde{c}_{2}^{\prime}$ and $d_{2}^{\prime}$

In what follows, for the sake of brevity, we will sometimes abuse notation and blur the distinction between a surface or a curve in $\Sigma_{2} \times \Sigma_{3}$ and its image under $q$ in the quotient manifold $X$. We will use the notation $[g, h]=g h g^{-1} h^{-1}$ for the commutator.

Lemma 7. Let $X_{0}$ denote the complement of tubular neighborhoods of the six Lagrangian tori of (7) in $X$. The Lagrangian framings give the following bases for the images of the fundamental groups of the 3-torus components of the boundary $\partial X_{0}$.

$$
\begin{array}{cc}
\left\{a_{1}, c_{1} ;\left[b_{1}^{-1}, d_{1}^{-1}\right]\right\}, & \left\{b_{1}, d_{1} c_{1} d_{1}^{-1} ;\left[a_{1}^{-1}, d_{1}\right]\right\}, \\
\left\{a_{2}, c_{1} ;\left[b_{2}^{-1}, d_{1}^{-1}\right]\right\}, & \left\{b_{2} a_{2} b_{2}^{-1}, d_{1} ;\left[b_{2}, c_{1}^{-1}\right]\right\},  \tag{10}\\
\left\{b_{1}, d_{2} ; a_{1}^{-1} a_{2}^{-1} \tilde{c}_{2}^{-1} a_{1} a_{2} \tilde{c}_{2}\right\}, & \left\{\tilde{c}_{2} b_{1} b_{2} \tilde{c}_{2}^{-1}, \tilde{c}_{2} b_{1} ;\left[a_{2}, \tilde{c}_{2} d_{2}^{-1} \tilde{c}_{2}^{-1}\right]\right\} .
\end{array}
$$

Proof. The point $q\left(\left\{z_{0}\right\} \times\left\{w_{0}\right\}\right)$ lies in $X_{0}$ and we choose it for the base point of $\pi_{1}\left(X_{0}\right)$. The first four triples in (10) are in standard form and can be derived as in [8]. For the fifth triple corresponding to the $q\left(b_{1}^{\prime} \times d_{2}^{\prime}\right)$ torus, the Lagrangian push-offs of $b_{1}^{\prime}$ and $d_{2}^{\prime}$ represent $b_{1}$ and $d_{2}$, respectively, by a standard argument in [8]. The orientation convention for the boundary $\partial X_{0}$ dictates that the third member of our triple (after the semicolon) should be the 'clockwise' meridian of $q\left(b_{1}^{\prime} \times d_{2}^{\prime}\right)$. (The anti-clockwise meridian is usually reserved for the boundary of the tubular neighborhood of $q\left(b_{1}^{\prime} \times d_{2}^{\prime}\right)$.) Note that the $q\left(b_{1}^{\prime} \times d_{2}^{\prime}\right)$ torus intersects the $q\left(\left(\tilde{a}_{1} \tilde{a}_{2}\right) \times \tau_{2}\left(\tilde{c}_{2}\right)\right)$ torus once negatively in $X$. Hence the clockwise meridian of $q\left(b_{1}^{\prime} \times d_{2}^{\prime}\right)$ is given by a word for the punctured $q\left(\left(\tilde{a}_{1} \tilde{a}_{2}\right) \times \tau_{2}\left(\tilde{c}_{2}\right)\right)$ torus, read in the anti-clockwise direction. To quickly reach the $b_{1}^{\prime} \times d_{2}^{\prime}$ torus from the preimage of the base point $\left\{z_{0}\right\} \times\left\{w_{0}\right\}$, we need to travel negatively in the $a_{1} \times\left\{w_{0}\right\}$ direction and negatively in the $\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}\right)$ direction. As explained in [8], the clockwise meridian of $q\left(b_{1}^{\prime} \times d_{2}^{\prime}\right)$ is then given by $q\left(\left(a_{2} a_{1}\right)^{-1} \tilde{c}_{2}^{-1}\left(a_{1} a_{2}\right) \tilde{c}_{2}\right)$, coming from the punctured $q\left(\left(\tilde{a}_{1} \tilde{a}_{2}\right) \times \tau_{2}\left(\tilde{c}_{2}\right)\right)$ torus. See the left half of Figure 4 .


Figure 4. Punctured $q\left(\left(\tilde{a}_{1} \tilde{a}_{2}\right) \times \tau_{2}\left(\tilde{c}_{2}\right)\right)$ and $q\left(a_{1} \times \zeta\right)$ tori
For the sixth triple, it is clear that the Lagrangian push-offs of $q(\beta)$ and $q(\xi)$ (see (8) and (9)) represent the homotopy classes of

$$
q\left(\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}\right)\right) \cdot\left(b_{1} \times\left\{w_{0}\right\}\right)\right)=q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right) \cdot q\left(b_{1} \times\left\{w_{0}\right\}\right)
$$

and

$$
\begin{aligned}
& q\left(\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}\right)\right) \cdot\left(b_{1} \times\left\{w_{0}\right\}\right) \cdot\left(b_{2} \times\left\{w_{0}\right\}\right) \cdot\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}^{-1}\right)\right)\right) \\
& =q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right) \cdot q\left(b_{1} \times\left\{w_{0}\right\}\right) \cdot q\left(b_{2} \times\left\{w_{0}\right\}\right) \cdot q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}^{-1}\right)
\end{aligned}
$$

respectively, in $\pi_{1}\left(X_{0}\right)$. Since $\beta$ and $\xi$ both start at $\left\{y_{0}\right\} \times\left\{v_{1}\right\}$, the lift of our meridian must start at the nearest preimage of the basepoint $\left\{z_{0}\right\} \times\left\{w_{1}\right\}$, rather than starting at $\left\{z_{0}\right\} \times\left\{w_{0}\right\}$. (Note that $\left\{z_{0}\right\} \times\left\{w_{0}\right\}$ lies near $\left\{y_{0}\right\} \times\left\{v_{0}\right\}$.) For analogy, note that for each of the first five triples, the lifts of the three loops must all start at the same vertex of the octagon and the same vertex of the dodecagon that are nearest to the chosen starting point on the corresponding surgery torus. Furthermore, the pairs of vertices are different for different surgery tori.


Figure 5. Loop $\zeta$ in $\Sigma_{3}$
Let $\zeta$ be the oriented loop in $\Sigma_{3}$ drawn in Figure 5, which represents the lower left portion of the dodecagon in Figure 2. Note that $\left\{z_{0}\right\} \times \zeta$ represents $\tilde{c}_{2} d_{2}^{-1} \tilde{c}_{2}^{-1}$ in $\pi_{1}\left(X_{0}\right)$. (There is a based homotopy between these loops that is supported inside $q\left(\left\{z_{0}\right\} \times \Sigma_{3}\right) \subset X_{0}$.) The $\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}$ cylinder intersects the $a_{1} \times \zeta$ torus once negatively at $\left\{y_{1}\right\} \times\left\{v_{2}\right\}$, which is near $\left\{y_{1}\right\} \times\left\{v_{1}\right\}$ and $\left\{z_{0}\right\} \times\left\{w_{1}\right\}$. Thus a lift of the clockwise meridian of $q\left(\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}\right)$, that starts at $\left\{z_{0}\right\} \times\left\{w_{1}\right\}$, is given by a word for the punctured $a_{1} \times \zeta$ torus, read in the anti-clockwise direction. To quickly reach the $\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}$ cylinder from the point $\left\{z_{0}\right\} \times\left\{w_{1}\right\}$, we need to travel positively in the $a_{1} \times\left\{w_{1}\right\}$ direction and positively in the $\left\{z_{0}\right\} \times \zeta$ direction. Note that $q\left(a_{1} \times\left\{w_{1}\right\}\right)=q\left(a_{2} \times\left\{w_{0}\right\}\right)$. Hence a lift of the clockwise meridian of the $q\left(\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}\right)$ torus, that starts at $\left\{z_{0}\right\} \times\left\{w_{1}\right\}$, represents $a_{2} \zeta a_{2}^{-1} \zeta^{-1}=\left[a_{2}, \zeta\right]=$ $\left[a_{2}, \tilde{c}_{2} d_{2}^{-1} \tilde{c}_{2}^{-1}\right]$ in $\pi_{1}\left(X_{0}\right)$. See the right half of Figure 4.
Remark 8. It turns out that $\left[a_{2}, \tilde{c}_{2} d_{2}^{-1} \tilde{c}_{2}^{-1}\right]=\left[a_{2}, d_{2}^{-1}\right]$ in $\pi_{1}\left(X_{0}\right)$. See (12) in the proof of Theorem 10 below. Hence the clockwise meridian of the $q\left(\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}\right)$ torus also represents $\left[a_{2}, d_{2}^{-1}\right]$ in $\pi_{1}\left(X_{0}\right)$.

Let $n \geq 1$ and $p \geq 0$ be a pair of integers. Inside $X$, we perform the following six torus surgeries:

$$
\begin{align*}
\left(a_{1}^{\prime} \times c_{1}^{\prime}, a_{1}^{\prime},-n\right), & \left(b_{1}^{\prime} \times c_{1}^{\prime \prime}, b_{1}^{\prime},-1\right) \\
\left(a_{2}^{\prime} \times c_{1}^{\prime}, c_{1}^{\prime},-1\right), & \left(a_{2}^{\prime \prime} \times d_{1}^{\prime}, d_{1}^{\prime},-1\right)  \tag{11}\\
\left(b_{1}^{\prime} \times d_{2}^{\prime}, d_{2}^{\prime},-1 / p\right), & \left(\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}, \beta,-1\right)
\end{align*}
$$

Here, we are using the notation from $[8,1]$. For example, the first surgery is a $(-n)$-surgery on $q\left(a_{1}^{\prime} \times c_{1}^{\prime}\right)$ torus along $q\left(a_{1}^{\prime}\right)$ loop with respect to the Lagrangian framing in (10). The first and the fifth surgeries are Luttinger surgeries (cf. [17, 4]) when $n=1$ and $p \geq 1$, respectively. If $p=0$, then the fifth surgery is trivial, i.e., the surgery does not alter the 4-manifold. The other four surgeries with -1 coefficient are all Luttinger surgeries.

Let $M_{n}^{p}$ denote the resulting closed 4-manifold after the surgeries in (11). If $p \geq 1$, then we have

$$
b_{1}\left(M_{n}^{p}\right)=b_{1}(X)-6=0 \quad \text { and } \quad b_{2}\left(M_{n}^{p}\right)=b_{2}(X)-2 \cdot 6=2
$$

The intersection form of $M_{n}^{p}$ is even for every $n \geq 1$ and $p \geq 0$. Note that $M_{1}^{p}$ is a minimal symplectic 4-manifold for every $p \geq 0$.

## 4. Calculation of fundamental group

The point $q\left(\left\{z_{0}\right\} \times\left\{w_{0}\right\}\right)$ lies in $X_{0} \subset M_{n}^{p}$ and hence we will choose it for the base point of $\pi_{1}\left(M_{n}^{p}\right)$.

Lemma 9. $\pi_{1}\left(M_{n}^{p}\right)$ is generated by $a_{1}, b_{1}, a_{2}, b_{2}, c_{1}, d_{1}, \tilde{c}_{2}, d_{2}$. The following relations hold in $\pi_{1}\left(M_{n}^{p}\right)$ :

$$
\begin{gathered}
a_{2}=\tilde{c}_{2}^{-1} a_{1} \tilde{c}_{2}, \quad b_{2}=\tilde{c}_{2}^{-1} b_{1} \tilde{c}_{2}, \quad b_{1}=\tilde{c}_{2}^{-1} b_{2} \tilde{c}_{2} \\
{\left[b_{2}, d_{2}\right]=1, \quad\left[a_{1}^{-1} b_{1}^{-1} a_{2}, d_{2}\right]=1, \quad\left[a_{2}^{-1} b_{2}^{-1} a_{1}, d_{2}\right]=1} \\
{\left[b_{1}^{-1}, d_{1}^{-1}\right]^{n}=a_{1}, \quad\left[a_{1}^{-1}, d_{1}\right]=b_{1}} \\
{\left[b_{2}^{-1}, d_{1}^{-1}\right]=c_{1}, \quad\left[b_{2}, c_{1}^{-1}\right]=d_{1}} \\
{\left[a_{1}, c_{1}\right]=1, \quad\left[b_{1}, c_{1}\right]=1, \quad\left[a_{2}, c_{1}\right]=1, \quad\left[a_{2}, d_{1}\right]=1, \quad\left[b_{1}, d_{2}\right]=1} \\
{\left[a_{1}, b_{1}\right]\left[a_{2}, b_{2}\right]=1, \quad\left[c_{1}, d_{1}\right]\left[\tilde{c}_{2}, d_{2}\right]=1}
\end{gathered}
$$

Proof. By using Seifert-Van Kampen theorem, the generators of $\pi_{1}\left(M_{n}^{p}\right)$ can be determined from Lemmas 5 and 7. The first relation comes from (1), which continues to hold in $\pi_{1}\left(M_{n}^{p}\right)$ because the image of homotopy (2) is contained in $q\left(a_{2} \times \tau_{2}\left(\tilde{c}_{2}\right)\right)$, which lies inside $X_{0} \subset M_{n}^{p}$. For example, we see that $q\left(a_{2} \times \tau_{2}\left(\tilde{c}_{2}\right)\right)$ is disjoint from the fifth surgery torus $q\left(b_{1}^{\prime} \times d_{2}^{\prime}\right)$ in (11), since $a_{2} \times \tau_{2}\left(\tilde{c}_{2}\right)$ is disjoint from $\left(b_{1}^{\prime} \times d_{2}^{\prime}\right) \cup\left(\tau_{1}\left(b_{1}^{\prime}\right) \times \tau_{2}\left(d_{2}^{\prime}\right)\right)$ in $\Sigma_{2} \times \Sigma_{3}$. See Figures 2 and 3 .

The second relation comes from (3), which continues to hold in $\pi_{1}\left(M_{n}^{p}\right)$ because the image of homotopy (4) is contained in $q\left(b_{2} \times \tau_{2}\left(\tilde{c}_{2}\right)\right) \subset X_{0}$. The third relation can be written as

$$
\begin{aligned}
q_{*}\left(b_{1} \times\left\{w_{0}\right\}\right) & =q_{*}\left(\left(\left\{z_{0}\right\} \times \tilde{c}_{2}^{-1}\right) \cdot\left(b_{2} \times\left\{w_{0}\right\}\right) \cdot\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)\right) \\
& =q_{*}\left(\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}^{-1}\right)\right) \cdot\left(b_{1} \times\left\{w_{1}\right\}\right) \cdot\left(\left\{z_{0}\right\} \times \tau_{2}\left(\tilde{c}_{2}\right)\right)\right)
\end{aligned}
$$

The corresponding based homotopy is given by

$$
F_{3}(s, t)=\left\{\begin{array}{lll}
q\left(b_{1}(0) \times \tau_{2}\left(\tilde{c}_{2}(1-3 t)\right)\right) & \text { if } \quad 0 \leq t \leq s / 3 \\
q\left(b_{1}\left(\frac{3 t-s}{3-2 s}\right) \times \tau_{2}\left(\tilde{c}_{2}(1-s)\right)\right) & \text { if } \quad s / 3 \leq t \leq(3-s) / 3 \\
q\left(b_{1}(1) \times \tau_{2}\left(\tilde{c}_{2}(3 t-2)\right)\right) & \text { if } \quad(3-s) / 3 \leq t \leq 1
\end{array}\right.
$$

The image of $F_{3}$ is contained in $q\left(b_{1} \times \tau_{2}\left(\tilde{c}_{2}\right)\right)$, which in turn lies inside $X_{0}$.
The fourth relation comes from the torus $q\left(b_{2} \times d_{2}\right)$ lying in $X_{0}$. The fifth relation comes from the torus $q\left(\gamma \times d_{2}\right)$ lying in $X_{0}$, where $\gamma$ is the closed path drawn in Figure 2. Note that

$$
q_{*}\left(\gamma \times\left\{w_{0}\right\}\right)=q_{*}\left(\left(a_{1}^{-1} \times\left\{w_{0}\right\}\right) \cdot\left(b_{1}^{-1} \times\left\{w_{0}\right\}\right) \cdot\left(a_{2} \times\left\{w_{0}\right\}\right)\right)
$$

via a based homotopy that is supported inside $q\left(\Sigma_{2} \times\left\{w_{0}\right\}\right) \subset X_{0}$. Similarly, the sixth relation comes from the torus $q\left(\delta \times d_{2}\right)$ in $X_{0}$, where $\delta$ is the closed path
drawn in Figure 2 satisfying

$$
q_{*}\left(\delta \times\left\{w_{0}\right\}\right)=q_{*}\left(\left(a_{2}^{-1} \times\left\{w_{0}\right\}\right) \cdot\left(b_{2}^{-1} \times\left\{w_{0}\right\}\right) \cdot\left(a_{1} \times\left\{w_{0}\right\}\right)\right)
$$

The next eleven relations are standard and can be derived as in [8] from Lemma 7 and the definition of torus surgery. Note that $q_{*}\left(\left[a_{1}, b_{1}\right]\left[a_{2}, b_{2}\right] \times\left\{w_{0}\right\}\right)=1$ holds in $\pi_{1}\left(M_{n}^{p}\right)$ because of the presence of the genus 2 surface $q\left(\Sigma_{2} \times\left\{w_{0}\right\}\right)$ inside $X_{0} \subset M_{n}^{p}$. The last relation $q_{*}\left(\left\{z_{0}\right\} \times\left[c_{1}, d_{1}\right]\left[\tilde{c}_{2}, d_{2}\right]\right)=1$ holds in $\pi_{1}\left(M_{n}^{p}\right)$ since $q\left(\left\{z_{0}\right\} \times \Sigma_{3}\right)$ is a genus 2 surface in $X_{0} \subset M_{n}^{p}$ whose fundamental group is generated by $q\left(\left\{z_{0}\right\} \times c_{1}\right)$, $q\left(\left\{z_{0}\right\} \times d_{1}\right), q\left(\left\{z_{0}\right\} \times \tilde{c}_{2}\right)$ and $q\left(\left\{z_{0}\right\} \times d_{2}\right)$.

Theorem 10. We have $\pi_{1}\left(M_{n}^{p}\right) \cong \mathbb{Z} / p$. In particular, $\pi_{1}\left(M_{n}^{0}\right) \cong \mathbb{Z}$ and $\pi_{1}\left(M_{n}^{1}\right)=$ 0 for every integer $n \geq 1$.
Proof. From the fifteenth and the sixteenth relations in Lemma 9, we know that $a_{2}$ commutes with both $c_{1}$ and $d_{1}$. Using the last relation in Lemma 9 , we deduce that $a_{2}$ commutes with $\left[\tilde{c}_{2}, d_{2}\right]=\left[c_{1}, d_{1}\right]^{-1}$. This implies that

$$
\begin{aligned}
1 & =\left[a_{2},\left[\tilde{c}_{2}, d_{2}\right]\right]=a_{2} \tilde{c}_{2} d_{2} \tilde{c}_{2}^{-1}\left(d_{2}^{-1} a_{2}^{-1} d_{2}\right) \tilde{c}_{2} d_{2}^{-1} \tilde{c}_{2}^{-1} \\
& =a_{2} \tilde{c}_{2} d_{2} \tilde{c}_{2}^{-1} a_{2}^{-1}\left[a_{2}, d_{2}^{-1}\right] \tilde{c}_{2} d_{2}^{-1} \tilde{c}_{2}^{-1} .
\end{aligned}
$$

Solving for $\left[a_{2}, d_{2}^{-1}\right]$, we conclude that

$$
\begin{equation*}
\left[a_{2}, d_{2}^{-1}\right]=\left(a_{2} \tilde{c}_{2} d_{2} \tilde{c}_{2}^{-1} a_{2}^{-1}\right)^{-1}\left(\tilde{c}_{2} d_{2}^{-1} \tilde{c}_{2}^{-1}\right)^{-1}=\left[a_{2}, \tilde{c}_{2} d_{2}^{-1} \tilde{c}_{2}^{-1}\right] \tag{12}
\end{equation*}
$$

Combining (12) with the twelfth relation in Lemma 9, we get $\left[a_{2}, d_{2}^{-1}\right]=\tilde{c}_{2} b_{1}$ and

$$
\begin{equation*}
a_{2} d_{2}^{-1} a_{2}^{-1}=\tilde{c}_{2} b_{1} d_{2}^{-1} \tag{13}
\end{equation*}
$$

From the fifth and the sixth relations in Lemma 9, we deduce that $d_{2}^{-1}$ commutes with the product

$$
\left(a_{2}^{-1} b_{2}^{-1} a_{1}\right)\left(a_{1}^{-1} b_{1}^{-1} a_{2}\right)=a_{2}^{-1} b_{2}^{-1} b_{1}^{-1} a_{2}
$$

It follows that

$$
\begin{equation*}
\left[b_{2}^{-1} b_{1}^{-1}, a_{2} d_{2}^{-1} a_{2}^{-1}\right]=a_{2}\left[a_{2}^{-1} b_{2}^{-1} b_{1}^{-1} a_{2}, d_{2}^{-1}\right] a_{2}^{-1}=1 \tag{14}
\end{equation*}
$$

Using (13), we can rewrite (14) as

$$
1=\left[b_{2}^{-1} b_{1}^{-1}, \tilde{c}_{2} b_{1} d_{2}^{-1}\right]=b_{2}^{-1} b_{1}^{-1} \tilde{c}_{2} b_{1} d_{2}^{-1} b_{1} b_{2} d_{2} b_{1}^{-1} \tilde{c}_{2}^{-1} .
$$

Since $d_{2}$ commutes with both $b_{1}$ and $b_{2}$ by the fourth and the seventeenth relations, we deduce that

$$
1=b_{2}^{-1} b_{1}^{-1} \tilde{c}_{2} b_{1} b_{1} b_{2} b_{1}^{-1} \tilde{c}_{2}^{-1}=b_{2}^{-1} b_{1}^{-1} b_{2} b_{2} b_{1} b_{2}^{-1}
$$

Hence $b_{1}^{-1} b_{2}^{2} b_{1} b_{2}^{-2}=1$, and so $b_{1}$ commutes with $b_{2}^{2}$.
From the ninth and the tenth relations, we deduce that

$$
\begin{aligned}
c_{1} & =b_{2}^{-1} d_{1}^{-1} b_{2} d_{1}=b_{2}^{-1}\left[c_{1}^{-1}, b_{2}\right] b_{2}\left[b_{2}, c_{1}^{-1}\right] \\
& =b_{2}^{-1} c_{1}^{-1} b_{2} c_{1} b_{2} c_{1}^{-1} b_{2}^{-1} c_{1}=b_{2}^{-1} d_{1}^{-1} b_{2} b_{2} c_{1}^{-1} b_{2}^{-1} c_{1}
\end{aligned}
$$

Canceling the $c_{1}$ 's from both sides and then rearranging, we conclude that

$$
d_{1}=b_{2}^{2} c_{1}^{-1} b_{2}^{-2}
$$

Since $b_{1}$ also commutes with $c_{1}$ by the fourteenth relation, $b_{1}$ must commute with $d_{1}$. It follows that $a_{1}=\left[b_{1}^{-1}, d_{1}^{-1}\right]^{n}=1$. From $a_{1}=1$, we can easily deduce that all other generators are trivial except for $d_{2}$. By Lemma $6, d_{2}$ has order $p$ in $\pi_{1}\left(M_{n}^{p}\right)$ if $p$ is a positive integer.

Remark 11. At the moment, $M_{1}^{p}$ has the smallest Euler characteristic amongst all closed minimal symplectic 4-manifolds having fundamental group isomorphic to $\mathbb{Z} / p$.

## 5. Homeomorphism and Seiberg-Witten invariant

Throughout this section, let $p=1$. For every integer $n \geq 1, M_{n}^{1}$ is a closed simply connected spin 4-manifold having intersection form $H$ (see (5)) with a basis given by the homology classes of genus 2 surfaces $q\left(\Sigma_{2} \times\left\{w_{0}\right\}\right)$ and $q\left(\left\{z_{0}\right\} \times \Sigma_{3}\right)$. By Freedman's classification theorem in [9], $M_{n}^{1}$ is homeomorphic to $S^{2} \times S^{2}$ for every $n \geq 1$.
Theorem 12. The symplectic 4-manifold $M_{1}^{1}$ is homeomorphic but not diffeomorphic to $S^{2} \times S^{2}$.
Proof. From [16], we know that the symplectic Kodaira dimension is a diffeomorphism invariant. The rational ruled surface $S^{2} \times S^{2}$ has Kodaira dimension $-\infty$. $X$ is a minimal surface of general type and hence has Kodaira dimension 2. Since $M_{1}^{1}$ is the result of six Luttinger surgeries on $X$ and Luttinger surgeries preserve symplectic Kodaira dimension (cf. [12]), $M_{1}^{1}$ has Kodaira dimension 2 as well.

Let $A$ and $B$ denote the 2-dimensional cohomology classes of $M_{n}^{1}$ that are Poincaré dual to the homology classes of $q\left(\Sigma_{2} \times\left\{w_{0}\right\}\right)$ and $q\left(\left\{z_{0}\right\} \times \Sigma_{3}\right)$, respectively. Let

$$
S W_{M_{n}^{1}}: H^{2}\left(M_{n}^{1} ; \mathbb{Z}\right) \longrightarrow \mathbb{Z}
$$

denote the 'small perturbation' Seiberg-Witten invariant of $M_{n}^{1}$ (cf. Lemma 3.2 in [20]).
Theorem 13. $S W_{M_{n}^{1}}(L) \neq 0$ only when $L= \pm(2 A+2 B)$, and

$$
\left|S W_{M_{n}^{1}}( \pm(2 A+2 B))\right|=n
$$

Proof. Let $Z$ denote the symplectic 4-manifold obtained by performing the following five Luttinger surgeries on $X$ :

$$
\begin{gathered}
\left(b_{1}^{\prime} \times c_{1}^{\prime \prime}, b_{1}^{\prime},-1\right), \quad\left(a_{2}^{\prime} \times c_{1}^{\prime}, c_{1}^{\prime},-1\right), \quad\left(a_{2}^{\prime \prime} \times d_{1}^{\prime}, d_{1}^{\prime},-1\right) \\
\left(b_{1}^{\prime} \times d_{2}^{\prime}, d_{2}^{\prime},-1\right), \quad\left(\left(\tilde{b}_{1} \tilde{b}_{2}\right) \times \tilde{c}_{2}^{\prime}, \beta,-1\right)
\end{gathered}
$$

Note that these are five of the six surgeries in (11) with $p=1$. Hence we obtain $M_{n}^{1}$ by performing $\left(a_{1}^{\prime} \times c_{1}^{\prime}, a_{1}^{\prime},-n\right)$ surgery on $Z$. We have $e(Z)=4, \sigma(Z)=0$, $b_{1}(Z)=1, b_{2}(Z)=4$, and the intersection form of $Z$ is isomorphic to $2 H$ with a basis given by

$$
\begin{equation*}
\left(\left[a_{1} \times c_{1}\right],-\left[b_{1} \times d_{1}\right]\right), \quad\left(\left[\Sigma_{2} \times\left\{w_{0}\right\}\right],\left[\left\{z_{0}\right\} \times \Sigma_{3}\right]\right) \tag{15}
\end{equation*}
$$

As shown in $[8,1]$, our theorem will follow at once if we can prove that the SeibergWitten invariant of $Z$ is nonzero only on $\pm c_{1}(Z)$.

We abuse the notation slightly and let $A$ and $B$ also denote the Poincaré duals of $\left[\Sigma_{2} \times\left\{w_{0}\right\}\right]$ and $\left[\left\{z_{0}\right\} \times \Sigma_{3}\right]$, respectively, in $H^{2}(Z ; \mathbb{Z})$. If $S W_{Z}(L) \neq 0$, then by applying the adjunction inequality to four surfaces in (15), we conclude that $L=r A+s B$, where $r$ and $s$ are even integers satisfying $|r| \leq 2$ and $|s| \leq 2$. Since the dimension of the Seiberg-Witten moduli space for $L$ is nonnegative, we must have

$$
L^{2}=2 r s \geq 2 e(Z)+3 \sigma(Z)=8
$$

It follows that $r=s= \pm 2$, and $L= \pm(2 A+2 B)=\mp c_{1}(Z)$. By Taubes's theorem in [21], we know that $\left|S W_{Z}\left( \pm c_{1}(Z)\right)\right|=1$.

Since the value of the Seiberg-Witten invariant for the canonical class of a symplectic 4-manifold is always $\pm 1$ by the work of Taubes [21] (see the proof of Theorem 1.2 in [20] for the $b_{2}^{+}=1$ case), $M_{n}^{1}$ cannot be symplectic when $n \geq 2$. Hence we conclude that $\left\{M_{n}^{1} \mid n \geq 2\right\}$ are irreducible (see Lemma 1), nonsymplectic and mutually nondiffeomorphic. This concludes the proof of Theorem 3.

Remark 14. In [19], Rasmussen has computed the Ozsváth-Szabó invariant of $M_{n}^{1}$, and has shown that $M_{n}^{1}$ 's do not admit any perfect Morse function.

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