Quivers

and their

Inverse Semigroup Algebras

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(Joint work with F. Gourdeau)

Dramatis Personae (Characters)

$S = \mathbb{Z}_+$ $\quad \mathcal{B}$ $\quad \mathcal{C}_n$ $\quad$ Quivers $\quad$ Buildings

$A = \mathbb{C}(S)$ algebraic $\quad \ell^1(S)$ discrete $\quad L^1(S)$ locally compact $\quad A(S)$ operator algebra

S path semigroup $\quad SS^*$ inverse semigroup $\quad E(SS^*)$ idempotents of inverse semigroup.

commutative semi lattice $e^2 = e$ $ef = fe$

HH^n simplicial (co)homology $\quad HC^n$ cyclic (co)homology $\quad A \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ general homology amenable? homological dimension?

HH^0 traces $\tau(ab) = \tau(ba)$

HH^1 $D(fg)(h) = D(f)(gh) + D(g)(hf)$

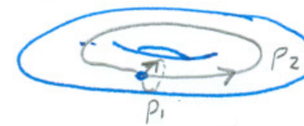
$\tau_D(fg) = D(fg)(1) = D(f)(g) + D(g)(f) = \tau_D(gf)$

K_0 $E, F \in M_n(A)$ $E + F = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & F \end{array}\right)$

$\tau(E+F) = \tau(E) + \tau(F)$

Dramatis Personae (Characters)

$S = \mathbb{Z}_+ \quad \mathcal{B} \quad \mathcal{C}_n$ Quivers Buildings

$A = \mathbb{C}(S) \quad \ell^1(S) \quad L^1(S) \quad A(S)$

algebraic — discrete — locally compact — operator algebra

$S \quad SS^* \quad E(SS^*)$

path semigroup — inverse semigroup — idempotents of inverse semigroup.

commutative semi lattice
$e^2 = e$
$ef = fe$

$HH^n \quad HC^n \quad A \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$

simplicial (co)homology — cyclic (co)homology — general homology

amenable?
homological dimension?

A Single Variable $\ell^1(\mathbb{Z}_+)$

$$\ell^1(\mathbb{Z}_+) = \left\{ f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n| = \|f\|_1 < \infty \right\}.$$

This is a completion of $\mathbb{C}[z]$.

⌜ Algebraically $\mathbb{C}[z]$ has easy homology as

$$0 \leftarrow \mathbb{C}[z] \leftarrow \mathbb{C}[z] \otimes \mathbb{C}[w] \leftarrow \mathbb{C}[z] \otimes \mathbb{C}[w] \leftarrow 0$$

$$P(z,z) \longleftarrow\!\shortmid P(z,w)$$

$$(z-w)Q(z,w) \longleftarrow\!\shortmid Q(z,w)$$

The corresponding diagram for $\ell^1(\mathbb{Z}_+)$ is not exact. ⌟

Cohomology 101

A a (Banach) algebra.
Y a (Banach) A-bi-module.

Derivations $D: A \to Y$

$$D(ab) = a.Db + Da.b$$

eg1 $A = \ell^1(\mathbb{Z}_+)$ $Y = \mathbb{C}_0$ character at 0.

$Df = f'(0)$ is a derivation.

eg2 $Y = A \hat{\otimes} A$ (recall $\mathbb{C}[z] \otimes \mathbb{C}[w]$)

$$D(a) = a \otimes 1 - 1 \otimes a$$

$$D(ab) = ab \otimes 1 - 1 \otimes ab$$

$$a\,Db + Da.b = a(b \otimes 1 - 1 \otimes a) + (a \otimes 1 - 1 \otimes a)b \quad \checkmark$$

Inner Derivations $D_y(a) = y.a - a.y$

Cohomology 201

Higher Cohomology

$$(\delta y)(f) = fy - yf \qquad \text{inner}$$

$$(\delta D)(f,g) = f.Dg - D(fg) + Df.g$$

$\delta D = 0 \iff D$ derivation

$$(\delta \varphi)(f,g,h) = f.\varphi(g,h) - \varphi(fg,h) + \varphi(f,gh) - \varphi(f,g)h$$

if $\varphi = \delta D$ then $\delta\varphi = 0$

even if D not a derivation.

φ is a 2-cocycle if $\delta\varphi = 0$

2-coboundary if $\varphi = \delta\psi$ some ψ.

eg $A = \ell^1(\mathbb{Z}_+^2)$ $\quad Y = \mathbb{C}_0$ so $f(z,w).\lambda = f(0,0)\lambda$.

$$\varphi(f,g) = \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}\right)\Bigg|_{(0,0)}$$

Cyclic Cohomology HC'

Recall simplicial cohomology $HH^n(A) = H^n(A; A^*)$

Some simplicial cohomology is better behaved.

We say D is a cyclic derivation if additionally

$$D(f)(g) = -D(g)(f).$$

eg 1 $HC'(\ell^1(\mathbb{Z}_+)) = 0.$ (HC' is cyclic version)

$$D(z^n)(z^m) = n\,D(z^1)(z^{n+m-1}) = \frac{n}{n+m} D(z^{n+m})(1) \quad \left(= \frac{n}{n+m} \tau_D(z^{n+m}). \right)$$

is cyclic

$$= \frac{n}{n+m} D(1)(z^{n+m}).$$

BUT derivations 'vanish' on idempotents (like 1!)

$$De = D(e^2) = e.De + De.e \text{ "="} \; 2e\,De.$$

$$\Rightarrow \quad e\,De = 2e\,De = 0$$

$$\Rightarrow \quad De = 0 \quad //$$

(More generally $\exists\, y \in Y$ s.t. $(D - \delta_y)(e) = 0$)

Cylic Cohomology HC^2

We say that a simplicial 2-cocycle is cyclic if

$$\varphi(f,g)(h) = +\varphi(g,h)(f)$$

when defining HC^2 we ask whether $\varphi = \delta\psi$ for some cyclic ψ i.e. is $\psi(f)(g) = -\psi(g)(f)$.

eg 2 Given a trace τ on A set.

$$\varphi_\tau(f,g)(h) = \tau(fgh) = \tau(ghf)$$
$$= \varphi_\tau(g,h)(f).$$

and easy to see $\delta\varphi_\tau = 0$.

Can φ_τ cobound (cyclically)?

Now for any idempotent $e \in A$.

$$\tau(e) = \varphi(e,e)(e) = (\delta\psi)(e,e)(e) = \psi(e)(e^2) - \psi(e^2)(e) + \psi(e)(e^2)$$
$$= \psi(e)(e) = -\psi(e)(e) = 0.$$

so any trace gives a cyclic 2-cocycle which cannot cobound on any algebra with a 1!

Quivers Quickly

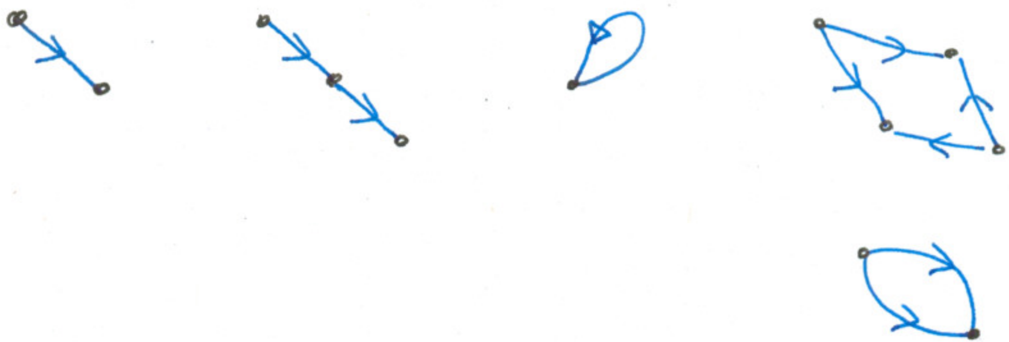
Quivers are a species of digraph
(directed graph)

They are (usually) taken to have a <u>finite</u> numbers of vertices and edges. and be connected.

BUT

They are allowed to have 1) vertex self loops
2) multiple edges

eg

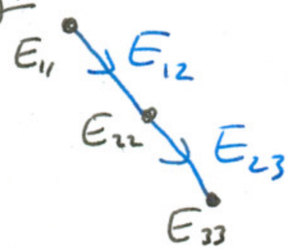


Path algebras of Quivers

Take an idempotent e_v for each vertex v.
Take a generator p_i for each edge.

NB 1) ends often do not specify an edge.
2) e_v is not a self loop.

eg1



path algebra $\begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{pmatrix}$

Upper triangular matrices T_3.
(a 0's and X's algebra).

eg2

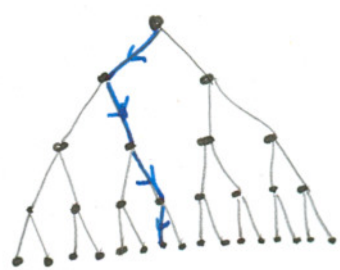
$z^5 =$ 

path algebra $\ell^1(\mathbb{Z}_+)$
(or $\mathbb{C}[z]$ or $A(D)$)

eg3

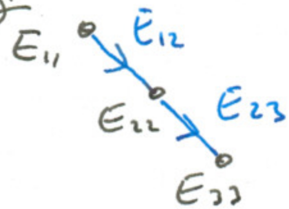
$\gamma = p_0 p_1 p_1 p_0 \in \mathbb{F}S_2$

path algebra $\ell^1(FS_2)$



Simplicial Derivations on Quiver algebras

eg1



$$\begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{pmatrix}$$

'wlog' $D(E_{ii}) = 0$.

$$D(E_{ij}) = D(E_{ii} E_{ij} E_{jj}) = E_{ii} D(E_{ij}) E_{jj}$$

$$\therefore \quad D(E_{ij}) = \alpha_{ij} E_{ij}$$

Note:

$$D(E_{13}) = D(E_{12} E_{23}) = D(E_{12}).E_{23} + E_{12} D(E_{23})$$

$$\alpha_{13} E_{13} = \alpha_{12} E_{12} E_{23} + \alpha_{23} E_{12} E_{23}$$

$$\Rightarrow \quad \alpha_{13} = \alpha_{12} + \alpha_{23}.$$

which is a 1-chain on the simplicial complex.

But $D(E_{ij})(E_{as}) = D(E_{ij})(E_{jj} E_{as} E_{ii}) = 0$. $HH^1 = 0$

eg2

As above $D(z^n)(z^m) = \frac{n}{n+m} D(z^{n+m})(1)$

$= \frac{n}{n+m} \tau_D(z^{n+m})$

but $\tau_D(1) = 1$ so $HH^1 \cong \ell^\infty(\mathbb{N})$.

Cyclic Cohomology of Quines algebras

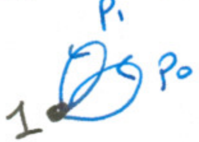
eg1 $HC^2(\ell^1(\mathbb{Z}_+)) \cong \mathbb{C}$

we have already seen $\varphi_\tau(f,g)(h) = \tau(fgh)$
is non-vanishing for $\tau = \varepsilon_0$ say.

In fact these are all 'equal'
and there are no others.

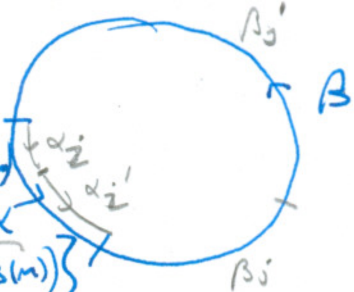
In general $HC^{odd}(\ell^1(\mathbb{Z}_+)) \cong 0.$
$HC^{even}(\ell^1(\mathbb{Z}_+)) \cong \mathbb{C}$ (even > 0.)

eg2 $HC^1(\ell^1(FS_2)) \cong 0$



Let $\alpha, \beta \in \ell^1(FS_2)$

D a cyclic derivation.



Set

$$\Psi(\alpha\beta) = \frac{1}{|\alpha\beta|}\Big\{ \sum_{i=1}^{|\alpha|}{}' D(\overbrace{\alpha(1)\dots\alpha(i)}^{\alpha_i})(\overbrace{\alpha(i+1)\dots\alpha(n)}^{\alpha_i'}\beta(1)\dots\beta(m)) + \sum_{j=1}^{|\beta|}{}' D(\alpha\overbrace{\beta(1)\dots\beta(j)}^{\beta_j})(\overbrace{\beta(j+1)\dots\beta(m)}^{\beta_j'})\Big\}$$

$$= \frac{1}{|\alpha\beta|}\Big\{\sum_i{}' -D(\alpha_i'\beta)(\alpha_i) + \sum_j{}'\big[D(\alpha)(\beta_j\beta_j') + D(\beta_j)(\beta_j'\alpha)\big]\Big\}$$

$$= \frac{1}{|\alpha\beta|}\Big\{\sum_i{}'\big[+D(\alpha_i')(\beta\alpha_i) + D(\beta)(\alpha_i\alpha_i')\big] + \sum_j{}' D(\alpha)(\beta) + \sum_j D(\beta_j)(\beta_j'\alpha)\Big\}$$

$$= \frac{1}{|\alpha\beta|}\Big\{\sum_i{}' D(\beta\alpha_i)(\alpha_i') - \sum_i{}' D(\beta)(\alpha) + \sum_j D(\alpha)(\beta) + \sum_j{}' D(\beta_j)(\beta_j'\alpha)\Big\}$$

$$= \frac{1}{|\alpha\beta|}\Big\{\sum_i{}' D(\alpha)(\beta) + \sum_j{}' D(\alpha)(\beta)\Big\} + \Psi(\beta\alpha)$$

$$\therefore\ D(\alpha)(\beta) = \Psi(\alpha\beta) - \Psi(\beta\alpha).$$

(HC^n is 'similar')

Connes-Tzygan Long Exact Sequence

$$0 \to HC^1 \xrightarrow[\text{inclusion}]{} HH^1 \to HC^0 \to HC^2 \xrightarrow[\text{inclusion}]{} HH^2 \to HC^1 \to HC^3$$

$\tau_D(f) = D(f)(1)$

$\varphi_\tau(f,g)(h) = \tau(fg)(h)$

$D_\varphi(f)(g) = \varphi(f,g)(1) - \varphi(g,f)(1)$

is an exact sequence,
we can get information about HC^* from HH^*
and vice versa.

eg $\ell^1(FS_n)$ $n = 1, 2, \ldots$

$$HC^{odd} = 0.$$

$$HC^{even} = \mathbb{C} \quad (\text{even} > 0)$$

and all isomorphic via
$\varphi_\tau(f_1, f_2, f_3, f_4)(f_0) = \tau(f_1 f_2 f_3 f_4 f_0)$

eg $\ell^1(FS_n)$ $\qquad n = 1, 2, \ldots$

$$HC^{odd} = 0$$

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and all isomorphic via

$$\varphi_\tau(f_1, f_2, f_3, f_4)(f_0) = \tau(f_1 f_2 f_3 f_4 f_0)$$

it follows from

$$HC^n \xrightarrow{0} HH^n \xrightarrow{0} HC^{n-1}$$

that $\quad HH^n \cong 0 \quad$ for $n \geq 2$.

for $\underline{\underline{n=1}}$

$$0 \to HH^1 \to HC^0 \to \mathbb{C} \to 0$$

$$0 \to HH^1 \to A^* \to \mathbb{C} \to 0$$

$$D \longmapsto \tau_D$$

$$HH^1 \cong (A/\mathbb{C})^*$$

Inverse Semigroup of a Quiver $\ell'(QQ^*)$

Let Q be a quiver and Q^* the reversed quiver.
Consider path α, β beginning at the same vertex.
The elements of the semigp QQ^* are pairs

(α, β^*)

α, β "paths" in Q starting together
β^* the reversed path of β.

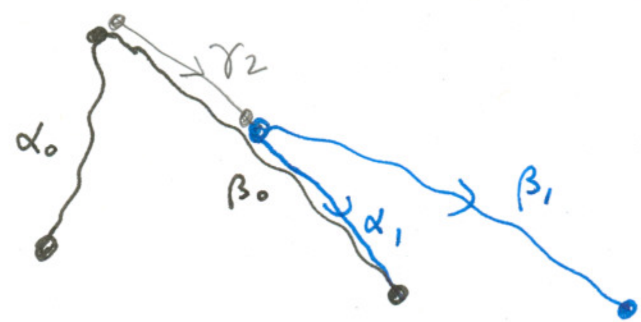
We define a product by

$$(\alpha_0, \beta_0^*) \circ (\alpha_1, \beta_1^*) = \begin{cases} (\alpha_0\gamma_1, \beta_1^*) & \text{if } \alpha_1 = \beta_0\gamma_1 \\ (\alpha_0, \gamma_2^*\beta_1^*) & \text{if } \beta_0 = \alpha_1\gamma_2 \\ 0 & \text{else.} \end{cases}$$

This leads to $\gamma^*\gamma = e_{\gamma(a)}$

N.B. The paths may be just an idempotent.

eg



Inverse Semigroup Examples

eg1

$$\begin{tikzcd} E_{11} \arrow[r, "E_{12}"] & E_{22} \arrow[r, "E_{23}"] & E_{33} \end{tikzcd}$$

$$\begin{pmatrix} \times & \times & \times \\ * & \times & \times \\ * & * & \times \end{pmatrix} \qquad M_3(\mathbb{C})$$

Is an amenable, contractible algebra with trivial cohomology.

Formally

$$E_{12} \sim (E_{12}, E_{22}^*)$$
$$E_{21} \sim (E_{22}, E_{12}^*)$$
$$E_{11} \sim (E_{11}, E_{11}^*)$$

eg2

$$\begin{tikzcd} 1 \arrow[loop right, "p"] \end{tikzcd}$$

$\mathcal{B}$ Bicyclic algebra.

$\mathcal{B} = \{ p^i q^j : i,j \geq 0 \} \qquad qp = 1.$

eg2 

$\mathcal{B}$ Bicyclic algebra.

$\mathcal{B} = \{ p^i q^j : i,j \geq 0 \}$ $\quad qp = 1.$

Theorem [GW] $\quad HH^n(\ell^1(\mathcal{B})) \cong HH^n(\ell^1(\mathbb{Z}_+)) \oplus HH^n(\ell^1(\mathbb{Z}_+))$

$\langle p^i \rangle \cong \mathbb{Z}_+ \qquad \langle q^j \rangle \cong \mathbb{Z}_+.$

Proof via a certain resolution of $\ell^1(\mathcal{B})$ by biprojective modules.

Compare

Toeplitz C^*-algebra generated by right shift on $\ell^2(\mathbb{Z}_+)$ and its adjoint.

$$0 \to K \to A \to C(\mathbb{T}) \to 0$$

compacts amenable $\qquad$ Toeplitz $\qquad$ $C(S^1)$ amenable.

So A is amenable. $\quad HH^1(\ell^1(\mathcal{B})) \cong \ell^\infty(\mathbb{N}) \oplus \ell^\infty(\mathbb{N})$

but $\ell^1(\mathcal{B})$ is not amenable.

The Cuntz Algebra $\ell^1(\mathcal{C}_2)$

The corresponding inverse semigroup has generators. p_1, p_2, q_1, q_2 with relations $q_i p_j = \delta_{ij} 1$.

$$\mathcal{C}_2 = \{ \gamma_1 \gamma_2^* : \gamma_1 \in FS_2 \ \gamma_2^* \in FS_2^* \} \cup \emptyset$$

Quiver Inverse Semigroups

<u>Theorem</u> [GW] For any quiver Q.

$$HC^n(\ell^1(QQ^*)) \cong \begin{cases} \mathbb{C} & n \text{ even} > 0 \\ 0 & n \text{ odd.} \end{cases}$$

<u>Cor</u> For any quiver Q

$$HH^n(\ell^1(QQ^*)) = 0 \qquad n \geq 2.$$

<u>Compare</u>

For any quiver Q the corresponding Cuntz-Krieger type algebra $C^*(Q)$ is amenable and hence $HH^n(C^*(Q)) = 0 \qquad n \geq 1$