THE INDEX FOR FREDHOLM ELEMENTS IN A BANACH ALGEBRA VIA A TRACE

JJ Grobler and H Raubenheimer

Definition 1 *Let I be an ideal in a Banach algebra. A function $\tau : I \to \mathbb{C}$ is called a* **trace** *if*

1. $\tau(p) = 1$ *for every rank one idempotent* $p \in I$.
2. $\tau(a + b) = \tau(a) + \tau(b)$ *for all* $a, b \in I$.
3. $\tau(\alpha a) = \alpha\tau(a)$ *for all* $\alpha \in \mathbb{C}$ *and* $a \in I$.
4. $\tau(ab) = \tau(ba)$ *for all* $a \in I$ *and* $b \in A$.

Definition 2 *Let A be a Banach algebra and I an ideal in A. We call an element $a \in A$* **a Fredholm element relative to I** *if there exists $a_0 \in A$ such that*

1. *$aa_0 - 1 \in I$ and*
2. *$a_0 a - 1 \in I$.*

The collection of all Fredholm elements relative to I will be denoted by $\Phi(A, I)$.

Definition 3 *Let* τ *be a trace on an ideal* I *in* A*. We define the* **index function** $\iota : \Phi(A, I) \to \mathbb{C}$ *by*

$$\iota(a) = \tau(aa_0 - a_0 a) \quad \textit{for all} \quad a \in \Phi(A, I).$$

where a_0 *satisfies* $aa_0 - 1 \in I$ *and* $a_0 a - 1 \in I$*.*

Proposition 1 *Let I be a trace ideal and $a \in \Phi(A, I)$.*

1. *The index function is well defined on $\Phi(A, I)$.*

2. *For $a, b \in \Phi(A, I)$, $\iota(ab) = \iota(a) + \iota(b)$.*

3. *For every $q \in I$ we have $\iota(a + q) = \iota(a)$.*

4. *For every $\lambda \neq 0$ and $q \in I$, $\iota(\lambda - q) = 0$.*

5. *The set $\Phi(A, I)$ is open in A.*

6. *The index function ι is constant on every component of $\Phi(A, I)$.*

7. *The index function $\iota : \Phi(A, I) \to \mathbb{C}$ is continuous.*

$$\mathsf{kh}(I) = \{a \in A \,|\, a + \bar{I} \in \mathsf{Rad}(A/\bar{I})\}$$

Theorem 1 *Let A be a semisimple Banach algebra and let the trace ideal I satisfy* $\mathsf{Soc}A \subset I \subset \mathsf{kh}(\mathsf{Soc}A)$. *Then*

1. $a \in \Phi(A, I)$ *if and only if there exists* $a_0 \in A$ *and idempotents* $p, q \in \mathsf{Soc}A$ *such that* $aa_0 = 1 - p$ *and* $a_0 a = 1 - q$.

2. $\iota(a) = \tau(q) - \tau(p)$.

3. $\iota(a) = n(a) - d(a)$.

If $a, b \in A$, then

$$\sigma(ab) \backslash \{0\} = \sigma(ba) \backslash \{0\}.$$

$$\varepsilon(ab) \backslash \{0\} = \varepsilon(ba) \backslash \{0\}?$$

Exp$A$ donotes the component of A^{-1} that contains 1. It is an open and closed subgroup of A^{-1}. The **exponential spectrum** of $a \in A$ is the set

$$\varepsilon(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \mathrm{Exp}A\}.$$

$\partial\varepsilon(a) \subset \sigma(a) \subset \varepsilon(a)$.

Theorem 2 (Murphy) *Let A be a Banach algebra and $a, b \in A$. Then each of the following conditions implies that $\varepsilon(ab)\backslash\{0\} = \varepsilon(ba)\backslash\{0\}$:*

1. *Either a or b is a limit or invertible elements.*

2. *A is of topological stable rank one.*

Let I be a closed trace ideal such that $\mathrm{Soc}A \subset I \subset \mathrm{kh}(\mathrm{Soc}A)$. Then

$$\Phi(A, I) = \cup_{n=-\infty}^{\infty} \iota^{-1}(n).$$

Let $J : A \to A/I$ be the natural homomorphism, ie, $J(x) = x + I$ for all $x \in A$.

Proposition 2 *Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with* $\mathrm{Soc}A \subset I \subset \mathrm{kh}(\mathrm{Soc}A)$. *Then*

$$J^{-1}\mathrm{Exp}(A/I) \subset \iota^{-1}(0).$$

Theorem 3 *Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}A \subset I \subset \mathrm{kh}(\mathrm{Soc}A)$. If $J^{-1}\mathrm{Exp}(A/I) = \iota^{-1}(0)$ then for all $a, b \in A$*

$$\varepsilon(ab + I)\backslash\{0\} = \varepsilon(ba + I)\backslash\{0\}.$$

Theorem 4 *Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}A \subset I \subset \mathrm{kh}(\mathrm{Soc}A)$. If $a, b \in A$ with $1 - ab \in \Phi(A, I)$, then $\iota(1 - ab) = \iota(1 - ba)$.*