G. Racher (Salzburg, Austria) 12 slides

On some injective group modules

1. All modules over an amenable algebra are amenable;

2. Injectivity of a reflexive $L^1(G)$-module implies G amenable;

3. An open problem.

Key words and phrases:

amenable Banach algebra,
injective and amenable modules,
reflexive Banach spaces,
weakly compact sets of vector measures,
variation of a vector measure,
amenable group.

1. All modules over an amenable algebra are amenable

Let A be a Banach algebra. A Banach left A-module is called injective if,

given any injective morphism $\iota : Y_0 \longrightarrow Y$ of Banach left A-modules admitting a bounded linear left inverse ℓ,

any left A-morphism λ_0 from Y_0 into X extends to a left A-morphism λ from Y into X:

$$
\begin{tikzcd}
Y_0 \arrow[r, "\iota"] \arrow[d, "\lambda_0"'] & Y \arrow[r, "\ell"] \arrow[dl, dashed, "\lambda"] & Y_0 \\
X & &
\end{tikzcd}
$$

$\iota, \lambda_0, \lambda$ A-morphisms,
ℓ only bdd linear,
$\ell \circ \iota = id_{Y_0}$
$\lambda \circ \iota = \lambda_0$.

Analogously: injectivity of right and A-bimodules.

A left A-module X is called amenable if its dual right A module X^* is injective.

Amenable modules $=$ flat modules of Helemskii

Amenable modules $\neq$ amenable unitary rep. of groups, cf B. Bekka, Invent. math. 100 (1992).

Lemma (A.Ya. Helemskii – M.V. Sheĭnberg). – All modules over an amenable algebra are amenable.

Proof. Let X be a left module over amenable A. Consider right A-morphisms ι and $\lambda_0 : Y_0 \longrightarrow X^*$. Then $\lambda_0 \circ \ell \in$ $\in \mathcal{L}(Y, X^*)$ and

$$D : A \longrightarrow \mathcal{L}(Y, X^*), \quad Da = a(\lambda_0 \circ \ell) - (\lambda_0 \circ \ell)a, \qquad a \in A,$$

where

$$[a(\lambda_0 \circ \ell)](y) = (\lambda_0 \circ \ell)(ya), \quad [(\lambda_0 \circ \ell)a](y) = (\lambda_0 \circ \ell)(y).a, \qquad y \in Y.$$

Da, $a \in A$, annihilates $\iota Y_0 \subset Y$:

$$\begin{aligned}
(Da)(\iota y) &= [a(\lambda_0 \circ \ell)](\iota y) - [(\lambda_0 \circ \ell)a](\iota y) \\
&= (\lambda_0 \circ \ell)((\iota y)a) - (\lambda_0 \circ \ell)(\iota y).a \\
&= (\lambda_0 \circ \ell)(\iota(ya)) - (\lambda_0 \circ \ell)(\iota y).a \\
&= \lambda_0(ya) - \lambda_0(y).a \\
&= 0 \qquad\qquad (y \in Y_0),
\end{aligned}$$

s.t.

$$\tilde{D} : A \longrightarrow \mathcal{L}(Y/\iota Y_0, X^*), \quad (\tilde{D}a)(\pi y) = (Da)(y), \quad y \in Y,$$

well-defined, $\pi : Y \longrightarrow Y/\iota Y_0$ can. projection.

$Y/\iota Y_0 \hat{\otimes} X$ becomes an A-bimodule by

$$a(\pi y \otimes x) = \pi y \otimes ax, \qquad (\pi y \otimes x)a = \pi ya \otimes x$$

and $\tilde{D}$ a derivation with values in $(Y/\iota Y_0 \hat{\otimes} X)^* = \mathcal{L}(Y/\iota Y_0, X^*)$

s.t. for some $S \in \mathcal{L}(Y/zY_0, X^*)$, $\tilde{D}a = a.S - S\cdot a$, $a \in A$,

$$a(\lambda_0 \circ \ell) - (\lambda_0 \circ \ell)a = a(S \circ \pi) - (S \circ \pi)a$$

i.e.

$$a(\lambda_0 \circ \ell - S \circ \pi) = (\lambda_0 \circ \ell - S \circ \pi)a, \qquad a \in A,$$

and

$$\lambda = \lambda_0 \circ \ell - S \circ \pi$$

is a right A-morphismus satisfying $\lambda \circ z_0 = \lambda_0$.

In case X is an A-bimodule, z and λ_0 are bimorphisms. Continuing with $\lambda \in \mathcal{L}_A(Y, X^*)$ above, we define

$$D_1 : A \longrightarrow \mathcal{L}_A(Y, X^*), \qquad D_1 a = a \bullet \lambda - \lambda \bullet a, \qquad a \in A,$$

where

$$(a \bullet \lambda)(y) = a . \lambda(y), \qquad (\lambda \bullet a)(y) = \lambda(ay), \qquad y \in Y.$$

$D_1 a$, $a \in A$, vanishing again on $zY_0 \subset Y$, we obtain

$$\tilde{D}_1 : A \longrightarrow \mathcal{L}_A(Y/zY_0, X^*) = \left[(Y/zY_0 \hat{\otimes} X)/K\right]^*,$$

where

$$(\tilde{D}_1 a)(\pi y) = (D_1 a)(y), \qquad K = \left[\pi y a \otimes x - \pi y \otimes a x\right], \; y \in Y, a \in A.$$

$\tilde{D}_1$ becomes a derivation with values in the dual A-bimodule $\mathcal{L}_A(Y/zY_0, X^*)$ s.t. for some $T \in \mathcal{L}_A(Y/zY_0, X^*)$

$$a \bullet \lambda - \lambda \bullet a = a \bullet (T \circ \pi) - (T \circ \pi) \bullet a, \qquad a \in A.$$

We finally see that

$$\tilde{\lambda} = \lambda_0 \circ \ell - S \circ \pi - T \circ \pi$$

is an A-bimorphism satisfying $\tilde{\lambda} \circ \imath = \lambda_0$. ■

Problem (A. Ya. Helemskii). — Let A be a Banach algebra all of whose Banach left A-modules are amenable. Does this imply the amenability of A?

2. Reflexivity of injective modules over $A = L^1(G)$

G locally compact group

B.E. Johnson (1972). —

$L^1(G)$ amenable algebra $\Longleftrightarrow$ G amenable group ■

A.Ya. Helemskii (VII.2.35). —

G amenable $\Longleftrightarrow$ $\mathbb{C}$ amenable (= injective) $L^1(G)$-module ■

H.G. Dales, M. Daws, H.L. Pham, P. Ramsden (2010). —

G amenable $\Longleftrightarrow$ $L^p(G)$, $1 < p < \infty$, amenable (= injective) left $L^1(G)$-module ■

Proposition. – G is amenable if and only if G admits a (non-trivial) injective Banach left $L^1(G)$-module which is reflexive as a Banach space.

Proof of Proposition

If G is amenable, then $L^1(G)$ is amenable s.t. all $L^1(G)$-modules are amenable. —

Let, conversely, G be arbitrary and X be an injective left $L^1(G)$-module, with $L^1(G)\cdot X \neq 0$ and X reflexive. We may assume $X = L^1(G)\cdot X$, s.t. X is a continuous isometric G-module

$$\|sx\| = \|x\| \qquad (s \in G,\ x \in X).$$

First part.

All the maps below have been introduced by Paul Ramsden (Będlewo, 2009). We consider

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "id_X = \lambda_0"'] & L^\infty(G)\check{\otimes} X \arrow[r, "\ell"] \arrow[dl, dashed] & X \\ X & & \end{tikzcd}$$

$L^\infty(G)\check{\otimes} X$ injective tensor product of Banach sp.

$$s(\varphi \otimes x) = L_s\varphi \otimes sx, \qquad s \in G,\ \varphi \in L^\infty(G),\ x \in X$$

$$\iota(x) = 1_G \otimes x, \qquad x \in X$$

$$\ell(\varphi \otimes x) = \left(\int \varphi(t) a(t)\, dt\right) x, \qquad \left(a \in L^1(G),\ \int a(t)\, dt = 1\right)$$

Then $\lambda_0, \imath$ are G-morphisms, and ℓ is bdd. linear s.t. $\ell \circ \imath = id_X$.

If X is **injective**, then there exists λ

$$X \xrightarrow{\ \imath\ } L^\infty(G) \check{\otimes} X \overset{\lambda}{\dashrightarrow} X$$

with the following properties

(i) λ bdd linear;

(ii) $\lambda(L_s \varphi \otimes sx) = s\,\lambda(\varphi \otimes x)$;

(iii) $\lambda(1_G \otimes x) = (\lambda \circ \imath)(x) = x$ $\qquad (s \in G, \varphi \in L^\infty(G), x \in X)$.

How to go on?

We define for any pair $(x, x^*) \in X \times X^*$, $\langle x, x^* \rangle = 1$, an element

$$\lambda_{x,x^*} \in L^\infty(G)^*, \qquad \lambda_{x,x^*}(\varphi) = \langle \lambda(\varphi \otimes x), x^* \rangle, \qquad \varphi \in L^\infty(G).$$

Then λ_{x,x^*} is a bdd linear functional on $L^\infty(G)$, and

$$\lambda_{x,x^*}(1_G) = \langle \lambda(1_G \otimes x), x^* \rangle = \langle x, x^* \rangle = 1.$$

Of course, λ_{x,x^*} need not be left translation invariant. But the set of its left translates,

$$\{L_s^* \lambda_{x,x^*} : s \in G\} \subset L^\infty(G)^*,$$

is at least invariant under the group of linear isometries $L_s^* : L^\infty(G)^* \longrightarrow L^\infty(G)^*$, $s \in G$. If we knew it to be rel. weakly, i.e. $\sigma(L^\infty(G)^*, L^\infty(G)^{**})$ - compact, Ryll-Nardzewski's fixed point theorem would furnish an element $M \in L^\infty(G)^*$ with

$$M(1_G) = 1, \qquad L_s^* M = M \qquad \forall s \in G,$$

implying the amenability of G.

Let us compute, for $\varphi \in L^\infty(G)$,

$$\begin{aligned} L_s^* \lambda_{x,x^*}(\varphi) &= \lambda_{x,x^*}(L_s\varphi) \\ &= \langle \lambda(L_s\varphi \otimes x), x^* \rangle \\ &= \langle \lambda(L_s\varphi \otimes s s^{-1} x), x^* \rangle \\ &= \langle s\lambda(\varphi \otimes s^{-1}x), x^* \rangle \\ &= \langle \lambda(\varphi \otimes s^{-1}x), x^* s \rangle \\ &= \lambda_{s^{-1}x,\, x^* s}(\varphi), \end{aligned}$$

s.t.

$$\{ L_s^* \lambda_{x,x^*} : s \in G \} = \{ \lambda_{s^{-1}x,\, x^* s} : s \in G \}.$$

Since $\| s^{-1} x \| = \| x \|$ and $\| x^* s \| = \| x^* \|$, $s \in G$, it will suffice to prove that

$$\{ \lambda_{x,x^*} : \|x\| \le 1,\ \|x^*\| \le 1 \} \subset L^\infty(G)^*$$

is relatively weakly compact in $L^\infty(G)^*$, if X is reflexive.

This is a purely linear affaire. Since $L^\infty(G) = C(K)$, for some compact Hd space, it will be enough to prove

Second part.

Main Lemma. – Let K be a compact Hd space, X a reflexive Banach space, and

$$\lambda : C(K) \check{\otimes} X \longrightarrow X$$

a bounded linear map. Then the set

$$\{ \lambda_{x, x^*} : \|x\| \leq 1, \|x^*\| \leq 1 \} \subset C(K)^*$$

is relatively weakly compact. $\square$

We shall consider the dual map $\lambda^* : X^* \longrightarrow (C(K) \check{\otimes} X)^*$. X^* being reflexive, we have $\lambda^*(\circ X^*) \subset (C(K) \check{\otimes} X)^*$ rel. weakly compact. To carry on we need a description of $(C(K) \check{\otimes} X)^*$.

Description of $(C(K) \check{\otimes} X)^*$.

$$\boxed{(C(K) \check{\otimes} X)^* = bvrca(\mathcal{B}(K), X^*)}$$

$$C(K) \xrightarrow{v} X^* \qquad\qquad \mathcal{B}(K) \xrightarrow{m} X^*$$

$$\begin{array}{l} \quad \nearrow \tilde{v} \\ C(K)^{**} \end{array}$$

$$m(A) = \tilde{v}(c_A), \qquad A \in \mathcal{B}orel(K)$$

m is then a regular countably additive measure of bounded variation (due to v being an integral linear map). We associate with $m : \mathcal{B}(K) \longrightarrow X^*$ a scalar measure, $|m|$, by

$$|m|(A) = \sup\left\{ \sum_{i=1}^{n} \|m(A_i)\| \;:\; A_i \in \mathcal{B}(K), \text{ pw. disj.}, A_i \subset A \right\},$$

for any $A \in \mathcal{B}(K)$, s.t. $|m|$ is a finite regular countably additive Borel measure on K,

$$|m|(K) < \infty .$$

Lemma 1. – K and X as before. Then we have:

$$\underset{\text{rel. weakly compact}}{C \subset (C(K) \check{\otimes} X)^*} \Longrightarrow \underset{\text{rel. weakly compact}}{|C| \subset C(K)^*} \qquad \blacksquare$$

Lemma 2 (Grothendieck, Canadian JM 5 (1953)). – Let D be a norm bounded subset of $C(K)^*$. Then D is relatively weakly compact if and only if for any sequence (A_n) of pw. disjoint Borel sets in K we have

$$\lim_n \mu(A_n) = 0 \qquad \text{uniformly for } \mu \in D. \qquad \blacksquare$$

Let $\lambda : C(K) \check{\otimes} X \longrightarrow X$ be bdd linear with X being reflexive. Then

(Lemma 1)
$$\lambda^*(\bigcirc X^*) \subset (C(K) \check{\otimes} X)^* \quad \text{rel. weakly comp.}$$
$$|\lambda^*(\bigcirc X^*)| \subset C(K)^* \quad \text{rel. weakly comp.}$$

Let (A_n) be a sequence of disjoint Borel sets in K, $x \in X$ and $x^* \in X^*$ of norm ≤ 1, then we have

$$\begin{aligned} |\lambda_{x,x^*}(A_n)| &= |\langle \lambda(c_{A_n} \otimes x), x^* \rangle| \qquad \text{(Slightly cheating)} \\ &= |\langle x, (\lambda^* x^*)^\sim (A_n) \rangle| \\ &\leq \|x\| \, \|(\lambda^* x^*)^\sim (A_n)\| \\ &\leq \|x\| \, |(\lambda^* x^*)^\sim| (A_n) \\ &\leq |(\lambda^* x^*)^\sim| (A_n) \\ &\leq \varepsilon \qquad (n \geq n_0) \end{aligned}$$

for some n_0, provided by Lemma 2. Again by Lemma 2, the set $\{\lambda_{x,x^*} : x \in \bigcirc X, x^* \in \bigcirc X^*\}$ is relatively weakly compact in $C(K)^*$, completing the proof of the Proposition. ∎

3. An open problem

Let $\mathcal{M}$ be a von Neumann algebra, admitting an injective normal Banach left $\mathcal{M}$-module X, reflexive as a Banach space. Does this imply the injectivity of $\mathcal{M}$?

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