A Multivariable Analogue of Ando's Theorem on Numerical Radius and C*-algebras with WEP

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Based on joint work with Douglas Farenick and Ali Kavruk
Builds on earlier work with Farenick, Kavruk, Ivan Todorov and Mark Tomforde

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A natural question to ask is if all the operators involved come from a given C*-algebra, then can the entries in the positive completion also come from that C*-algebra?

In the single operator case, we prove that this places no restriction on the C*-algebra. But as soon as one considers the two variable case, we prove that the entries can be chosen from the given C*-algebra iff the C*-algebra has WEP.

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Finally, it shows that Connes' Embedding Problem is equivalent to a 3×3 matrix completion problem.

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- Background needed for proofs:
- Define quotients in the category whose objects are operator systems, with morphisms the unital, completely positive maps.
- Farenick-P result that the operator system spanned by free unitaries is a quotient of tridiagonal matrices.
- Results follow by combining this fact with Kirchberg's characterization of WEP and earlier work on tensor products of operator systems by Kavruk-P-Todorov-Tomforde.

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Theorem (Ando)

Given $X \in B(H)$, we have that $w(X) \leq 1/2$ iff there exist $A_1, A_2 \in B(H)$ with $A_1 + A_2 = I$ such that

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is positive and invertible in $B(H \oplus H)$.

A Multivariable Analogue of Ando's Theorem

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Given $X_1, \ldots, X_n \in B(H)$, their **free joint numerical radius** is

$$w(X_1, \ldots, X_n) \equiv \sup\{w(X_1 \otimes U_1 + \cdots + X_n \otimes U_n) : U_1, \ldots, U_n \text{ are unitary}\}$$

where the supremum is taken over all n-tuples of unitaries on all Hilbert spaces and the tensor is spatial.

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where the supremum is taken over all n-tuples of unitaries on all Hilbert spaces and the tensor is spatial.
It is not hard to see that it is sufficient to replace "all" Hilbert spaces by a single, separable infinite dimensional Hilbert space and that the supremum is actually attained. Also this value remains the same if the unitaries are replaced by all contractions.

Theorem (1,FKP)

Let $X_1, \ldots, X_n \in B(H)$*. Then* $w(X_1, \ldots, X_n) < 1/2$

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$$\begin{bmatrix} A_1 & X_1 & 0 & \cdots & 0 \\ X_1^* & A_2 & X_2 & & \vdots \\ 0 & X_2^* & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_n & X_n \\ 0 & \cdots & 0 & X_n^* & A_{n+1} \end{bmatrix} \qquad (1)$$

is positive and invertible in $M_{n+1}(B(H)) = B(H^{(n+1)})$.

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is positive and invertible in $M_{n+1}(B(H)) = B(H^{(n+1)})$. Hence, $w(X_1, \ldots, X_n) = 1/2 \inf\{\|A_1 + \cdots + A_{n+1}\|\}$ over all A's satisfying (1).

The Weak Expectation Property

Recall that a C*-algebra $\mathcal{A}$ has WEP iff for every faithful representation $\pi : \mathcal{A} \to B(H)$ there exists a UCP idempotent map $E : B(H) \to \pi(\mathcal{A})''$, such that $E(\pi(a)) = \pi(a)$ for every $a \in \mathcal{A}$.

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Theorem (2,FKP)

Let $\mathcal{A} \subseteq B(H)$ be a unital C-subalgebra. Then $\mathcal{A}$ has WEP iff $\forall p \in \mathbb{N}$, $\forall X_1, X_2 \in M_p(\mathcal{A})$ such that $w(X_1, X_2) < 1/2$, the A_1, A_2, A_3 in equation (1) can be chosen from $M_p(\mathcal{A})$.*

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Note that since $w(X_1, X_2) < 1/2$ such operators exist in $M_p(B(H)) = B(H^{(p)})$ and WEP is equivalent to saying that whenever this positive completion problem can be solved "over B(H)" for elements of $\mathcal{A}$, then it can also be solved over $\mathcal{A}$.

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If $\mathcal{A}$ is a von Neumann algebra, then it is injective iff it has WEP, so the above also gives a matrix completion characterization of injectivity.

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If we let $\mathbb{F}_2$ denote the free group on two generators, then Kirchberg has shown that Connes' embedding problem is equivalent to deciding whether or not the full group C*-algebra $C^*(\mathbb{F}_2)$ has WEP.

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Theorem (3,FKP)

Fix a representation $C^*(\mathbb{F}_2) \subseteq B(H)$. Connes' embedding problem is true iff $\forall p \in \mathbb{N}$ whenever equation (1) can be solved over $M_p(B(H))$ for elements of $M_p(C^*(\mathbb{F}_2))$ then it can be solved over $M_p(C^*(\mathbb{F}_2))$.

Quotients of Operator Systems

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A subspace $\mathcal{K} \subset \mathcal{S}$ is a *kernel* if there are an operator system $\mathcal{T}$ and a completely positive linear map $\phi : \mathcal{S} \to \mathcal{T}$ such that $\mathcal{K} = \ker \phi$.

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If $\mathcal{K} \subset \mathcal{S}$ is a kernel, then define $C_n(\mathcal{S}/\mathcal{K}) \subset M_n(\mathcal{S}/\mathcal{K})_{\mathrm{sa}}$ to be

$$\{\dot{H} \, : \, \forall \, \varepsilon > 0 \; \exists \, K_\varepsilon \in M_n(\mathcal{K})_{\mathrm{sa}} \text{ such that } \varepsilon 1 + H + K_\varepsilon \in M_n(\mathcal{S})_+\} \, .$$

The collection $\{C_n(\mathcal{S}/\mathcal{K})\}_{n\in\mathbb{N}}$ is a family of cones that endow $\mathcal{S}/\mathcal{K}$ with the structure of an abstract operator system with (Archimedean) order unit $\dot{1} = q(1)$.

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The operator system $(\mathcal{S}/\mathcal{K}, \{C_n(\mathcal{S}/\mathcal{K})\}_{n\in\mathbb{N}}, q(1))$ *is called the* **quotient operator system**.

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Note: Complete order isomorphisms(COI) are the "natural" identifications in the category whose objects are operator systems and whose morphisms are the UCP maps.

Two Quotient Examples

Let $\mathcal{J}_n \subseteq M_n$ denote the set of diagonal matrices of trace 0. It may not be obvious that this is the kernel of a UCP map, but we shall show that it is shortly.

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Let $E_{i,j}$ denote the standard matrix units. For $i \neq j$, note that

$$dist(E_{i,j}, \mathcal{J}_n) = \inf\{\|E_{i,j} + K\| : K \in \mathcal{J}_n\} = 1.$$

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We now show that in the operator system quotient $\|E_{i,j} + \mathcal{J}_n\| \leq 1/n$!

Note that $E_{i,i} - E_{j,j} \in \mathcal{J}_n$. Hence, if ϕ is a UCP map with $\mathcal{J}_n = ker(\phi)$, then $\phi(E_{i,i}) = \phi(E_{j,j})$.

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But since, $\begin{pmatrix} E_{i,i} & E_{i,j} \\ E_{j,i} & E_{j,j} \end{pmatrix} \in M_2(M_n)^+$, we have that $\|\phi(E_{i,j})\| \leq 1/n$.

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The above calculation, leads one to wonder: What is $M_n/\mathcal{J}_n$?

Let $C^*(\mathbb{F}_{n-1})$ denote the full C*-algebra of the free group on $n-1$ generators, denoted $u_2, ..., u_n$, set $u_1 = 1$ and let $\mathcal{W}_{n-1} = span\{u_i u_j^*\} \subseteq C^*(\mathbb{F}_{n-1})$.

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Theorem (1,FP)

The map $\phi : M_n \to \mathcal{W}_{n-1}$ defined by $\phi(E_{i,j}) = \frac{u_i u_j^*}{n}$ is a complete quotient map, i.e., $M_n/\mathcal{J}_n$ and $\mathcal{W}_{n-1}$ are COI. Moreover, $C_e^*(M_n/\mathcal{J}_n) = C^*(\mathbb{F}_{n-1})$.

Lifting Tridiagonals and the WEP

Let $\mathcal{T}_n \subseteq M_n$ denote the set of tridiagonal matrices, and let $\mathcal{S}_{n-1} = span\{1, u_1, u_1^*, ..., u_{n-1}, u_{n-1}^*\} \subseteq C^*(\mathbb{F}_{n-1})$.

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Proposition (2,FP)

The UCP map $\psi : \mathcal{T}_n \to \mathcal{S}_{n-1}$, given by $\psi(E_{i,i}) = \frac{1}{n}$, $\psi(E_{i,i+1}) = \frac{u_i}{n}$ and $\psi(E_{i+1,i}) = \frac{u_i^*}{n}$ is a complete quotient map. Hence, $\mathcal{T}_n/\mathcal{J}_n = \mathcal{S}_{n-1}$ up to COI. Moreover, $C_e^*(\mathcal{T}_n/\mathcal{J}_n) = C^*(\mathbb{F}_{n-1})$.

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We use this quotient to define an "exactness" or "lifting property" of an operator system $\mathcal{R}$.

We say that $\mathcal{R}$ has **property** $(\mathfrak{S}_n)$ iff $\psi \otimes id_{\mathcal{R}} : \mathcal{T}_n \otimes_{min} \mathcal{R} \to (\mathcal{T}_n/\mathcal{J}_n) \otimes_{min} \mathcal{R}$ is a complete quotient map. We say $\mathcal{R}$ has **property** $(\mathfrak{S})$ when it has this property for all n.

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It is easy to see directly that $B(H)$ has property $(\mathfrak{S})$, since the matrices do.

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Corollary (Kirchberg)

$B(H) \otimes_{min} C^*(\mathbb{F}_n) = B(H) \otimes_{max} C^*(\mathbb{F}_n)$ for all n.

Combining with Kirchberg's characterization of WEP we have:

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Let $\mathcal{A}$ be a C-algebra. Then the following are equivalent:*

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Thus, WEP is equivalent to a question about positive liftings of tensors with 3×3 tridiagonal matrices.

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The proof of the converse uses the same diagram and the fact from FP that the max tensor preserves quotients.

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The multivariable analogue of Ando[1,FKP] follows from observing that for $X_1, \dots, X_n \in B(H)$ we have $w(X_1, \dots, X_n) < 1/2$ iff

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Using the fact that $B(H)$ has property $(\mathfrak{S})$ allows us to lift this to a strictly positive element of $B(H) \otimes_{min} \mathcal{T}_{n+1}$ and this lifting has the desired form.
When $X_1, X_2 \in M_p(\mathcal{A})$, then the ability to lift these special positive elements to positive elements of $M_p(\mathcal{A}) \otimes_{min} \mathcal{T}_{n+1}$ is enough to guarantee that $\mathcal{A}$ has property $(\mathfrak{S}_3)$ which is equivalent to WEP.

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Thus, $w(X) < 1/2 \Rightarrow I \otimes 1 + X \otimes u_1 + X^* \otimes u_1^* > 0 \Rightarrow$ lifts to a strictly positive element of $M_2(\mathcal{A})$ of the form given by Ando AND with $A_1, A_2 \in \mathcal{A}$.

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So the fact that we can do 1-variable Ando in the C*-algebra follows from the fact that $C(\mathbb{T})$ is nuclear. We show that, in fact, it is equivalent to the fact that $C(\mathbb{T})$ is nuclear.

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9. $\mathcal{U}_2 \otimes_{max} \mathcal{U}_2 \subseteq_{coi} \mathcal{V}_2 \otimes_{max} \mathcal{V}_2$,

where $\mathcal{E}_3, \mathcal{U}_2, \mathcal{V}_2$ *are certain spaces of matrices.*

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These arise because: $\mathcal{U}_2 = \mathcal{S}_2^d$, $\mathcal{V}_2 = \mathcal{T}_3^d$ and $\mathcal{E}_3 = \mathcal{W}_2^d$ are COI's.

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All on arxiv
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Thanks for your time!