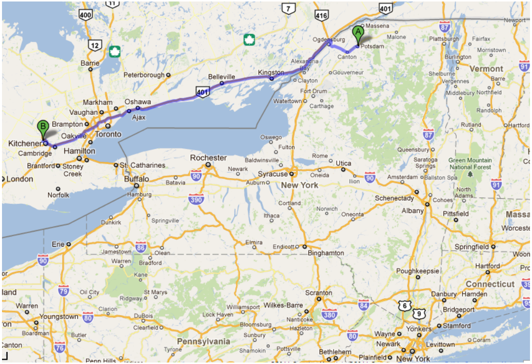
Some Further Results on Boundaries of Algebras of Lipschitz Functions



Aaron Luttman
Clarkson University

Banach Algebras
2011

Joint work with Kassie Averill, Ann Johnston, Ryan Northrup, and Robert Silversmith

Research supported by U.S. National Science Foundation under grant DMS-1004531 and the U.S. National Security Agency under grant H98230-09-1-0097.

Program Description

General Info

- 14 Undergraduate Students from U.S.A. and Mexico
- 4 Research "Teams" of 3 or 4 students, each with 1 faculty mentor (knot theory, quantum graph theory, dynamics in Banach spaces, and function algebras)
- Beginning of June to end of July
- Approximately 50 such programs are currently funded by the U.S. National Science Foundation and U.S. National Security Agency

Our Group

- Williams College (Massachusetts), Harvey Mudd College (California), SUNY Potsdam (New York)
- 2 Mathematics students and 1 Math Education student

Fundamental Goal: Can we develop a theory of Lipschitz algebras that mimics the theory of uniform algebras? Like $C(X)$ is well-understood, so is $\mathrm{Lip}(X)$, but, whereas subalgebras of $C(X)$ are also well-understood, the subalgebras of $\mathrm{Lip}(X)$ are considerably less developed (Weaver, 1999).

Fundamental Goal: Can we develop a theory of Lipschitz algebras that mimics the theory of uniform algebras? Like $C(X)$ is well-understood, so is $\mathrm{Lip}(X)$, but, whereas subalgebras of $C(X)$ are also well-understood, the subalgebras of $\mathrm{Lip}(X)$ are considerably less developed (Weaver, 1999).

Smaller-scale, Short-term Questions:

1) What is the structure/characterization of boundaries of algebras of Lipschitz functions?

2) Is the characterization the same for such algebras over $\mathbb{R}$ vs. over $\mathbb{C}$?

3) Does commutativity matter?

4) Can separation of points be assumed without loss of generality in this case?

Boundaries of Families of Functions

Definition Given a family of continuous functions $\mathcal{A}$ on a compact Hausdorff space X, a *boundary* for $\mathcal{A}$ is a set $B \subset X$ such that every function $f \in \mathcal{A}$ attains its maximum modulus on B.

- Classically studied for uniform algebras, i.e. Shilov and Choquet boundaries (Kaniuth, 2009).
- Theory is quite different for algebras over $\mathbb{C}$ than for algebras over $\mathbb{R}$ (Kulkarni and Limaye, 1992).
- For real-linear vector spaces of functions, this is closely related to the question of function extension.
- Can be studied for collections of functions without algebraic structure, but the applications are not as obvious (Lambert and L., 2011).

Some Notations and Motivations

Primary Goal for Project:

Understand and characterize the boundaries for algebras of Lipschitz functions taking values in $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (quaternions), in particular the boundaries that consist only of weak peak points.

Tools Needed:

1) $M(f)$ – the *maximizing set* of a function f, i.e.
$M(f) = \{x \in X : |f(x)| = \|f\|_\infty\}$

2) *m-set* – intersection of maximizing sets, i.e. E is an m-set if
$E = \bigcap_{f \in S} M(f)$ for some family of functions S

3) *weak peak point* – singleton m-sets, i.e. $\{x_0\} = \bigcap_{f \in S} M(f)$ for some family of functions S

Further Tools and Motivations

Notes:

• It can always be shown that there exist minimal m-sets with respect to inclusion.

• When the minimal m-sets are singletons – as is the case for uniform algebras – it can be shown that the intersection of all closed boundaries is the closure of the set of weak peak points (Lambert and L., 2011).

• We seek ways to prove that the minimal m-sets for a particular algebra must be singletons.

• For uniform algebras, this can be done using a classical result due to Bishop (1959).

• Using these constructions requires that the range in which the functions take their values have multiplicative norm.

Further Tools and Motivations

Notes:

• It can always be shown that there exist minimal m-sets with respect to inclusion.

• When the minimal m-sets are singletons – as is the case for uniform algebras – it can be shown that the intersection of all closed boundaries is the closure of the set of weak peak points (Lambert and L., 2011).

• We seek ways to prove that the minimal m-sets for a particular algebra must be singletons.

• For uniform algebras, this can be done using a classical result due to Bishop (1959).

• Using these constructions requires that the range in which the functions take their values have multiplicative norm.

Further Tools and Motivations

Notes:

• It can always be shown that there exist minimal m-sets with respect to inclusion.

• When the minimal m-sets are singletons – as is the case for uniform algebras – it can be shown that the intersection of all closed boundaries is the closure of the set of weak peak points (Lambert and L., 2011).

• We seek ways to prove that the minimal m-sets for a particular algebra must be singletons.

• For uniform algebras, this can be done using a classical result due to Bishop (1959).

• Using these constructions requires that the range in which the functions take their values have multiplicative norm.

Further Tools and Motivations

Notes:

• It can always be shown that there exist minimal m-sets with respect to inclusion.

• When the minimal m-sets are singletons – as is the case for uniform algebras – it can be shown that the intersection of all closed boundaries is the closure of the set of weak peak points (Lambert and L., 2011).

• We seek ways to prove that the minimal m-sets for a particular algebra must be singletons.

• For uniform algebras, this can be done using a classical result due to Bishop (1959).

• Using these constructions requires that the range in which the functions take their values have multiplicative norm.

Further Tools and Motivations

Notes:

• It can always be shown that there exist minimal m-sets with respect to inclusion.

• When the minimal m-sets are singletons – as is the case for uniform algebras – it can be shown that the intersection of all closed boundaries is the closure of the set of weak peak points (Lambert and L., 2011).

• We seek ways to prove that the minimal m-sets for a particular algebra must be singletons.

• For uniform algebras, this can be done using a classical result due to Bishop (1959).

• Using these constructions requires that the range in which the functions take their values have multiplicative norm.

Normed Algebras over $\mathbb{R}$

(Classical?) Theorem

Suppose that $\mathcal{A}$ is an associative, unital, normed algebra over $\mathbb{R}$ such that $\|fg\| = \|f\|\|g\|$ for all $f, g \in \mathcal{A}$. Then $\mathcal{A}$ is a division algebra. In particular, $\mathcal{A}$ is isometrically algebra isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Note: It is well-known (Hurwitz, 1898) that an associative, normed, division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (quaternions), so it is only necessary to show that an algebra $\mathcal{A}$ satisfying our condition is a division algebra.

Sketch of Convoluted Proof

Step 1

Theorem (Abel and Jarosz, 2003, Jarosz, 2008)
If $\mathcal{A}$ is a Banach algebra satisfying $\|f^2\| = \|f\|^2$ for all $f \in \mathcal{A}$, then there exists a compact Hausdorff space X such that $\mathcal{A}$ is isometrically, algebraically isomorphic to a subalgebra $\hat{\mathcal{A}}$ of $C(X, \mathbb{H})$.

Sketch of Convoluted Proof

Step 1

Theorem (Abel and Jarosz, 2003, Jarosz, 2008)
If $\mathcal{A}$ is a Banach algebra satisfying $\|f^2\| = \|f\|^2$ for all $f \in \mathcal{A}$, then there exists a compact Hausdorff space X such that $\mathcal{A}$ is isometrically, algebraically isomorphic to a subalgebra $\hat{\mathcal{A}}$ of $C(X, \mathbb{H})$.

Lemma
If $\mathcal{A}$ is an associative, unital, normed algebra over $\mathbb{R}$ such that $\|fg\| = \|f\|\|g\|$ for all $f, g \in \mathcal{A}$, then its function algebra representation $\hat{\mathcal{A}}$ contains only functions of constant real part.

Sketch of Convoluted Proof, Part 2

Step 2

- Since $\hat{\mathcal{A}}$ is unital, it contains the real constants.
- Since $\mathrm{Re}(\hat{f})$ is constant for every $f \in \mathcal{A}$, $\mathcal{A}$ contains $\mathrm{Re}(\hat{f})$ for every $f \in \mathcal{A}$, so $\overline{\hat{f}} = 2\mathrm{Re}(\hat{f}) - \hat{f} \in \hat{\mathcal{A}}$.
- Therefore $\hat{\mathcal{A}}$ contains $|f|^2 = \hat{f}\overline{\hat{f}}$, which implies that it contains $\hat{f}^{-1} = \frac{\overline{\hat{f}}}{|\hat{f}|^2}$ (for non-zero $\hat{f}$).
- Thus every non-zero element $\hat{f} \in \hat{\mathcal{A}}$ is invertible, making $\hat{\mathcal{A}}$ a division algebra.

Sketch of Convoluted Proof, Part 2

Step 2

- Since $\hat{\mathcal{A}}$ is unital, it contains the real constants.
- Since $\mathrm{Re}(\hat{f})$ is constant for every $f \in \mathcal{A}$, $\mathcal{A}$ contains $\mathrm{Re}(\hat{f})$ for every $f \in \mathcal{A}$, so $\overline{\hat{f}} = 2\mathrm{Re}(\hat{f}) - \hat{f} \in \hat{\mathcal{A}}$.
- Therefore $\hat{\mathcal{A}}$ contains $|f|^2 = \hat{f}\overline{\hat{f}}$, which implies that it contains $\hat{f}^{-1} = \frac{\overline{\hat{f}}}{|\hat{f}|^2}$ (for non-zero $\hat{f}$).
- Thus every non-zero element $\hat{f} \in \hat{\mathcal{A}}$ is invertible, making $\hat{\mathcal{A}}$ a division algebra.

Step 3 Invoke Hurwitz to get that $\mathcal{A} \cong \hat{\mathcal{A}} \cong \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

...Back to Boundaries

Theorem (Bishop, 1959)
Let $\mathcal{A}$ be a uniform algebra on X, $x_0 \in X$ a weak peak point, and $f \in \mathcal{A}$ such that $f(x_0) \neq 0$. Then there exists $h \in \mathcal{A}$ such that h and fh attain their maximum modulus at x_0, i.e. $x_0 \in M(h) \cap M(fh)$.

Proof relies fundamentally on uniform closure, but uniform algebras are not the only algebras in which this result holds.

...Back to Boundaries

Theorem (Bishop, 1959)
Let $\mathcal{A}$ be a uniform algebra on X, $x_0 \in X$ a weak peak point, and $f \in \mathcal{A}$ such that $f(x_0) \neq 0$. Then there exists $h \in \mathcal{A}$ such that h and fh attain their maximum modulus at x_0, i.e. $x_0 \in M(h) \cap M(fh)$.

Proof relies fundamentally on uniform closure, but uniform algebras are not the only algebras in which this result holds.

Theorem
If X is a compact metric space, $x_0 \in X$, and $f \in \mathrm{Lip}(X, \mathbb{F})$ such that $f(x_0) \neq 0$, then there exists $h \in \mathrm{Lip}(X, \mathbb{F})$ such that h and fh attain their maximum modulus exclusively at x_0, i.e. $M(fh) = M(h) = \{x_0\}$.

Initial Conjecture

Conjecture

If $\mathcal{A}$ is a norm-complete subalgebra of $\text{Lip}(X)$, $x_0 \in X$, and $f \in \mathcal{A}$ such that $f(x_0) \neq 0$, then there exists $h \in \mathcal{A}$ such that h and fh attain their maximum modulus exclusively at x_0, i.e. $M(fh) = M(h) = \{x_0\}$.

Initial Conjecture

Conjecture

If $\mathcal{A}$ is a norm-complete subalgebra of $\text{Lip}(X)$, $x_0 \in X$, and $f \in \mathcal{A}$ such that $f(x_0) \neq 0$, then there exists $h \in \mathcal{A}$ such that h and fh attain their maximum modulus exclusively at x_0, i.e. $M(fh) = M(h) = \{x_0\}$.

Counterexample

Let $C^1([0,1], \mathbb{R})$ be the continuously differentiable, real-valued functions on $[0,1]$. Then $C^1([0,1], \mathbb{R})$ is a complete subalgebra of $\text{Lip}([0,1], \mathbb{R})$, but it does not satisfy the conjecture, meaning there exists x_0's and f's for which no such h exists.

Continuously Differentiable Functions

Proposition

Let $x_0 \in (0,1)$ and $f \in C^1([0,1],\mathbb{R})$ be such that $f(x_0) \neq 0$. Then there exists $h \in C^1([0,1],\mathbb{R})$ such that $M(fh) = M(h) = \{x_0\}$ if and only if $f'(x_0) = 0$.

Continuously Differentiable Functions

Proposition

Let $x_0 \in (0,1)$ and $f \in C^1([0,1],\mathbb{R})$ be such that $f(x_0) \neq 0$. Then there exists $h \in C^1([0,1],\mathbb{R})$ such that $M(fh) = M(h) = \{x_0\}$ if and only if $f'(x_0) = 0$.

Proof

($\Rightarrow$) If $M(fh) = M(h) = \{x_0\}$, then $h(x_0) \neq 0$, $h'(x_0) = 0$, and

$$0 = (fh)'(x_0) = f'(x_0)h(x_0) + f(x_0)h'(x_0) = f'(x_0)h(x_0),$$

which implies that $f'(x_0) = 0$.

($\Leftarrow$) Construct exponential function h that does exactly what's needed. This direction does not generalize to subalgebras of $C^1([0,1],\mathbb{R})$.

What's the Problem?

1) Smoothness actually hurts; we need enough functions that have "corners."

2) How is a "corner" defined in general?

3) We don't really care if fh and h both maximize at x_0. What we really need is that fh maximizes there. Can we find such an h?

What's the Problem?

1) Smoothness actually hurts; we need enough functions that have "corners."

2) How is a "corner" defined in general?

3) We don't really care if fh and h both maximize at x_0. What we really need is that fh maximizes there. Can we find such an h?

What's the Problem?

1) Smoothness actually hurts; we need enough functions that have "corners."

2) How is a "corner" defined in general?

3) We don't really care if fh and h both maximize at x_0. What we really need is that fh maximizes there. Can we find such an h?

Corners

Definition. Let X be a compact metric space. Then we say that a Lipschitz function h has a *corner* at x_0 in the interior of X if and only if $h(x_0)$ is a local extremum and

$$\ell_{x_0}(h) := \sup_{\epsilon>0} \left\{ \inf_{0<d(x,x_0)<\epsilon} \frac{\left| |h(x)| - |h(x_0)| \right|}{d(x,x_0)} \right\} > 0.$$

Note: If $h\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz, then $\ell_{x_0}(h) = 0$ if and only if h is differentiable at x_0.

A Partial Characterization

Theorem

Let (X, d) be a compact metric space and $\mathcal{A}$ a complete, point-separating subalgebra of $\mathrm{Lip}(X, \mathbb{F})$. If for every weak peak point $x_0 \in X$ there exists $h \in \mathcal{A}$ such that $x_0 \in M(h)$ and

$$\ell_{x_0}(h) := \sup_{\epsilon>0} \left\{ \inf_{0<d(x,x_0)<\epsilon} \frac{\left| |h(x)| - |h(x_0)| \right|}{d(x, x_0)} \right\} > 0,$$

then for every $f \in \mathcal{A}$ with $f(x_0) \neq 0$ there exists $k \in \mathcal{A}$ such that $x_0 \in M(k)$ and $M(fk) = \{x_0\}$.

What this Means in Terms of Boundaries

Corollary

If $\mathcal{A} \subset \mathrm{Lip}(X, \mathbb{F})$ satisfies the hypotheses of the above theorem, then

1. every function $f \in \mathcal{A}$ is constant on every minimal *m*-set, so
2. all minimal *m*-sets are singletons (by separation of points),
3. every weak peak point is a strong peak point, which implies that
4. the intersection of all closed boundaries for $\mathcal{A}$ is a closed boundary for $\mathcal{A}$ and is the closure of the weak peak points.

What this Means in Terms of Boundaries

Corollary

If $\mathcal{A} \subset \text{Lip}(X, \mathbb{F})$ satisfies the hypotheses of the above theorem, then

1. every function $f \in \mathcal{A}$ is constant on every minimal *m*-set, so
2. all minimal *m*-sets are singletons (by separation of points),
3. every weak peak point is a strong peak point, which implies that
4. the intersection of all closed boundaries for $\mathcal{A}$ is a closed boundary for $\mathcal{A}$ and is the closure of the weak peak points.

Thanks!

References

M. Abel and K. Jarosz. "Noncommutative uniform algebras." *Studia Math.*, 162:213-218, 2004.

A. Hurwitz. "Über die composition der quadratischen formen von beliebig vielen variabeln." *Nachr. Ges. Wiss. Göttingen*, 1:309-316, 1898.

K. Jarosz. "Function representation of a noncommutative uniform algebra." *Proc. Amer. Math. Soc.*, 136:605-611, 2008.

E. Kaniuth. *A Course in Commutative Banach Algebras*, Volume 246 of Graduate Texts in Mathematics. Springer, 2009.

S. H. Kulkarni and B. V. Limaye. *Real Function Algebras*. Pure and Applied Mathematics. Marcel Dekker, Inc., 1992.

S. Lambert and A. Luttman. *Generalized Strong Boundary Points and Boundaries of Families of Continuous Functions*. *Mediter. J. Math.*, to appear.

N. Weaver. *Lipschitz Algebras*. World Scientific, 1999.