Commutative amenable operator algebras

Yemon Choi
University of Saskatchewan

Banach Algebras 2011
University of Waterloo, 9th August 2011

A motivating example

Consider $\ell^1(\mathbb{Z})$ with convolution product. This is a standard example of an amenable Banach algebra.

If $\mathcal{H}$ is a Hilbert space and $\phi : \ell^1(\mathbb{Z}) \to \mathcal{B}(\mathcal{H})$ is a bounded algebra homomorphism, then $\mathfrak{A} := \overline{\mathsf{ran}(\phi)}^{\|\cdot\|}$ is a commutative, amenable, operator algebra.

As a Banach algebra, $\mathfrak{A}$ is a familiar object in mild disguise...

A motivating example

Consider $\ell^1(\mathbb{Z})$ with convolution product. This is a standard example of an amenable Banach algebra.

If $\mathcal{H}$ is a Hilbert space and $\phi : \ell^1(\mathbb{Z}) \to \mathcal{B}(\mathcal{H})$ is a bounded algebra homomorphism, then $\mathfrak{A} := \overline{\mathsf{ran}(\phi)}^{\|\cdot\|}$ is a commutative, amenable, operator algebra.

As a Banach algebra, $\mathfrak{A}$ is a familiar object in mild disguise...

Theorem

Let ϕ be as above. There exists a positive invertible operator R such that $R\phi(\delta_g)R^{-1}$ is unitary for each $g \in \mathbb{Z}$.

That is: ϕ is similar to a *-representation. It follows that $\mathfrak{A}$ is isomorphic to a quotient algebra of $\mathrm{C}^*_r(\mathbb{Z} \cong C(\mathbb{T})$.

Remark

A more concrete formulation of the theorem: every doubly-power bounded operator on $\mathcal{H}$ is similar to a unitary operator.

Remark

A more concrete formulation of the theorem: every doubly-power bounded operator on $\mathcal{H}$ is similar to a unitary operator.

In the theorem, we can replace $\mathbb{Z}$ with any amenable group G (Dixmier; Day). As a very special case:

Theorem (various)

Let $\mathcal{F}$ be a **bounded** *family of commuting idempotents in $\mathcal{B}(\mathcal{H})$, such that if $E, F \in \mathcal{F}$ then $E + F - 2EF \in \mathcal{F}$. Then there exists R positive and invertible, such that RER^{-1} is self-adjoint for each $E \in \mathcal{F}$.*

Amenability gives us splitting

Amenable Banach algebras were introduced by Johnson (1972) and can be characterized as those Banach algebras possessing "bounded approximate diagonals".

Amenability gives us splitting

Amenable Banach algebras were introduced by Johnson (1972) and can be characterized as those Banach algebras possessing "bounded approximate diagonals".

Proposition

If A is amenable, and $A \to \mathcal{B}(\mathcal{H})$ is a bounded representation of A on Hilbert space, then each A-invariant closed subspace $V \subseteq \mathcal{H}$ has an A-invariant complement.

Corollary

The disc algebra is not amenable. (Consider its usual representation on $H^2(\mathbb{T}) \cong \ell^2(\mathbb{Z}_+)$.)

Amenable operator algebras?

A cheap way to obtain amenable operator algebras:

– take an amenable C*-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$

(e.g. $C(X)$, $\mathcal{K}(\mathcal{H})$, $\mathrm{C}^*_r(SL_2(\mathbb{R}))$, C^*_r(discrete amenable))

– take $R \in \mathcal{B}(\mathcal{H})$ which is invertible

– put $\mathfrak{A} = R^{-1}\mathcal{A}R$.

Amenable operator algebras?

A cheap way to obtain amenable operator algebras:

– take an amenable C*-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$

(e.g. $C(X)$, $\mathcal{K}(\mathcal{H})$, $\mathrm{C}_r^*(SL_2(\mathbb{R}))$, C_r^*(discrete amenable))

– take $R \in \mathcal{B}(\mathcal{H})$ which is invertible

– put $\mathfrak{A} = R^{-1}\mathcal{A}R$.

Does every amenable operator algebra arise in this way?

No complete answer known. I want to single out two partial results:

Theorem (Willis, 1995)

Let T be a **compact** *operator on Hilbert space. Then $\overline{\mathsf{alg}\,T}^{\|\cdot\|}$ is amenable if and only if T is similar to a (compact) normal operator.*

Theorem (Gifford, 1997/2006)

Let $\mathfrak{A}$ be a norm-closed, amenable subalgebra of $\mathcal{K}(\mathcal{H})$. Then $\mathfrak{A}$ is similar to a self-adjoint subalgebra.

Theorem (Willis, 1995)

Let T be a **compact** *operator on Hilbert space. Then $\overline{\mathsf{alg}\, T}^{\|\cdot\|}$ is amenable if and only if T is similar to a (compact) normal operator.*

Theorem (Gifford, 1997/2006)

Let $\mathfrak{A}$ be a norm-closed, amenable subalgebra of $\mathcal{K}(\mathcal{H})$. Then $\mathfrak{A}$ is similar to a self-adjoint subalgebra.

Beyond the compact case ... not much is known! Partial results: Curtis & Loy (1995), generalizing Sheinberg (1977); Marcoux (2008).

Finite von Neumann algebras

A von Neumann algebra $\mathcal{M}$ is said to be finite if it has a separating family of normal tracial states.

Examples of finite von Neumann algebras:

$L^\infty(\Omega)\,\overline{\otimes}\,\mathbb{M}_n$; $\mathrm{VN}(G)$ for G compact or discrete ; $\bigotimes_{n=1}^{\infty} \mathbb{M}_2$.

Finite von Neumann algebras

A von Neumann algebra $\mathcal{M}$ is said to be finite if it has a separating family of normal tracial states.

Examples of finite von Neumann algebras:

$L^\infty(\Omega)\,\overline{\otimes}\,\mathbb{M}_n$; $\mathrm{VN}(G)$ for G compact or discrete ; $\bigotimes_{n=1}^\infty \mathbb{M}_2$.

Theorem (C., submitted)

Let $\mathcal{M}$ be a finite von Neumann algebra, and $\mathfrak{A}$ a commutative, norm-closed, (operator) amenable subalgebra. Then there exists a positive invertible $R \in \mathcal{M}$, such that $R\mathfrak{A}R^{-1}$ is self-adjoint.

We shall sketch the ideas in the proof, at least in the case when $\mathcal{M}$ has a faithful normal, tracial state τ and $1_{\mathcal{M}} \in \mathfrak{A}$.

The proof is long with many a winding turn

By a result of Sheinberg (1977), the Gelfand representation $\mathsf{G} : \mathfrak{A} \to C(X)$ has dense range. The main work lies in showing G is bijective.

Injectivity of G is proved by adapting arguments from [Willis, 1995].

The proof is long with many a winding turn

By a result of Sheinberg (1977), the Gelfand representation $\mathsf{G} : \mathfrak{A} \to C(X)$ has dense range. The main work lies in showing G is bijective.

Injectivity of G is proved by adapting arguments from [Willis, 1995]. In fact we get a quantitative version.

Proposition

$\mathfrak{A} \hookrightarrow \mathcal{M}$ has a bounded linear extension $\theta : C(X) \to L^1(\mathcal{M}, \tau)$. More precisely,

$$\|a\|_{1,\tau} \leq Cr(a) \text{ for all } a \in \mathfrak{A}$$

where C = amenability constant of $\mathfrak{A}$, $r(\cdot)$ is the spectral radius, and $\|\cdot\|_{1,\tau}$ is the L^1-norm on $\mathcal{M}$ induced by τ.

We want to find idempotents in (or closely related to) $\theta(C(X))$, and drag them into $\mathcal{M}$ somehow.

We want to find idempotents in (or closely related to) $\theta(C(X))$, and drag them into $\mathcal{M}$ somehow.

To make sense of this, embed $L^1(\mathcal{M})$ continuously into a genuine semitopological algebra $\widetilde{\mathcal{M}}$. (This is the completion of $\mathcal{M}$ in the noncommutative measure topology defined by τ.)

Would like to work with $(\theta(\chi_U))_{U \, \mathrm{open}}$, but in general $\chi_U \notin C(X)$.

We want to find idempotents in (or closely related to) $\theta(C(X))$, and drag them into $\mathcal{M}$ somehow.

To make sense of this, embed $L^1(\mathcal{M})$ continuously into a genuine semitopological algebra $\widetilde{\mathcal{M}}$. (This is the completion of $\mathcal{M}$ in the noncommutative measure topology defined by τ.)

Would like to work with $(\theta(\chi_U))_{U\,\mathrm{open}}$, but in general $\chi_U \notin C(X)$.

To make this work: extend θ continuously to $\Theta : \widetilde{\mathsf{B}} \to L^1(\mathcal{M}, \tau)$, where $\widetilde{\mathsf{B}}$ is a function algebra containing all such χ_U. This can be done using an old result of Tomczak-Jaegermann (1974).

Hence $(\Theta(\chi_U))_U$ is a family of commuting idempotents in $\widetilde{\mathcal{M}}$, approximable by elements of $\mathfrak{A}$...

Adapting a trick of Gifford

Given $e \in \Theta(\widetilde{\mathsf{B}}) \subseteq \widetilde{\mathcal{M}}$ with $e^2 = e$, why should it live in $\mathcal{M}$, let alone satisfy $\|e\| \leq C$?

Well, e corresponds to a closed, densely-defined idempotent M_e on $\mathcal{H}$. By adapting the proof of a result from [Gifford,2006] for bounded idempotents, we can show M_e is **bounded** and $\|M_e\| \leq C$, where C is the amenability constant of $\mathfrak{A}$.

Adapting a trick of Gifford

Given $e \in \Theta(\widetilde{\mathsf{B}}) \subseteq \widetilde{\mathcal{M}}$ with $e^2 = e$, why should it live in $\mathcal{M}$, let alone satisfy $\|e\| \leq C$?

Well, e corresponds to a closed, densely-defined idempotent M_e on $\mathcal{H}$. By adapting the proof of a result from [Gifford,2006] for bounded idempotents, we can show M_e is **bounded** and $\|M_e\| \leq C$, where C is the amenability constant of $\mathfrak{A}$.

The basic idea: $\mathrm{Ran}(M_e)$ is a closed $\mathfrak{A}$-invariant subspace, so has an $\mathfrak{A}$-invariant complement. The corresponding bounded idempotent Q commutes with M_e and has the same range space, forcing $M_e = Q$.

Jigsaw falling into place

- For each open $U \subseteq X$, $\Theta(\chi_U)$ is an idempotent in $\widetilde{\mathcal{M}}$.
- By previous remarks, there exists $R \in \mathcal{M}$ such that each $R\Theta(\chi_U)R^{-1}$ is an orthogonal projection.
- Using approximation arguments, can show $R\Theta(\cdot)R^{-1}$ defines a *-homomorphism $\widetilde{\mathsf{B}} \to \mathcal{M}$.
- This implies that G is bounded below; we already knew it had dense range, so it is a bijection.

Obstacles to future progress?

So what can be done for an amenable operator algebra $\mathfrak{A}$?

Common theme of proofs to date results to date: we attempt to diagonalize elements of $\mathfrak{A}$.

We want a large supply of $\mathfrak{A}$-invariant subspaces of $\mathcal{H}$. (Each of these will have an $\mathfrak{A}$-invariant complement.)

Also desirable: a large number of idempotents in WOT-closure of $\mathfrak{A}$. But how do you produce these idempotents?

So perhaps we need to understand $\mathrm{Lat}(\mathfrak{A})$ better, or find some functional calculus to substitute for Borel FC. . .

Selected references

- YC, On commutative, operator amenable subalgebras of finite von Neumann algebras, *preprint*, arXiv `1012.4259`
- J. A. Gifford, Operator algebras with a reduction property. *J. Austr. Math. Soc.* **80** *(2006) 297–315*
- L. W. Marcoux, On abelian, triangularizable, total reduction algebras. *J. Lond. Math. Soc. (2)* **77** *(2008) 164–182*
- M. V. Šeĭnberg, A characterization of the algebra $C(\Omega)$ in terms of cohomology groups. *Uspekhi Mat. Nauk.* **32** *(1977) 203–204.*
- G. A. Willis, When the algebra generated by an operator is amenable. *J. Operator Theory* **34** *(1995) 239–249*