Common hypercyclic vectors for paths of hypercyclic operators

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Definition

An operator $T \in B(X)$ is *hypercyclic* if there is a vector $x \in X$ for which its orbit Orb$(T, x) = \{x, Tx, T^2x, T^3x, \ldots, T^nx, \ldots\}$ is dense in X.

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$\mathcal{HC}(T)$ = the set of all hypercyclic vectors for T, which is dense.

Definition

A bounded linear operator $T : \ell^2 \to \ell^2$ is a *unilateral weighted backward shift* if there is a bounded positive weight sequence $\{w_1, w_2, w_3, \ldots\}$ such that

$$T(a_0, a_1, a_2, a_3, \ldots) = (w_1 a_1, w_2 a_2, w_3 a_3, \ldots).$$

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Theorem (Rolewicz, 1969)

If $t > 1$, then the unilateral weighted backward shift tB is hypercyclic.

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$$\begin{aligned}\mathcal{HC}(T) &= \left\{x \in X : \forall i \geq 1, \forall k \geq 1, \exists n \text{ s. t. } T^n x \in B(x_i, \frac{1}{k})\right\} \\ &= \bigcap_{i,k\geq 1} \bigcup_{n\geq 1} T^{-n} B(x_i, \frac{1}{k}), \text{ which is a dense } G_\delta \text{ set.}\end{aligned}$$

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Hence if (T_n) is a countable family of hypercyclic operators, then the set $\bigcap \mathcal{HC}(T_n)$ of *common hypercyclic vectors* for (T_n) is also a dense G_δ set.

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Question

What about an uncountable family $\{T_t\}$ of hypercyclic operators?

Theorem (Abakumov and Gordon, 2003), (Costakis and Sambarino, 2004)

If B is the unilateral backward shift, then $\bigcap_{t>1} \mathcal{HC}(tB)$ is a dense G_δ set.

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Definition

A family of operators $\{F_t \in B(X) : t \in I\}$, where I is an interval, is a *path of operators* if the map $F : I \longrightarrow B(X)$, defined by

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is continuous.

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Reproof (2009)

Reestablished the above result with a simple proof, using the idea of paths.

Theorem (2009)

Between two hypercyclic unilateral weighted backward shifts,

(1) there is a path of such operators with a dense G_δ set of common hypercyclic vectors.

(2) there is a path of such operators with no common hypercyclic vector.

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- Observation: The more operators a path contains, the less likely that the path has a common hypercyclic vector.
- Question: Can a path be dense in a certain way, and yet the whole path still has a common hypercyclic vector?

Let $\mathcal{B}$ be the set of all unilateral weighted backward shifts on ℓ^2.

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Proposition (2011)

There is a path in $\mathcal{B}$ which is SOT-dense in $\mathcal{B}$, and every operator along the path has the exact same dense G_δ set of hypercyclic vectors.

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Definition

A *hypercyclic subspace* for T is a closed, infinite dimensional subspace of X consisting entirely, except for the zero vector, of hypercyclic vectors.

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There is a path of hypercyclic shifts in $\mathcal{B}$ that is SOT-dense in $\mathcal{B}$ such that the whole path has a common hypercyclic subspace.

For a separable, infinite dimensional Banach space X, let $T \in B(X)$, and

$$\mathcal{S}(T) = \{L^{-1}TL : L \text{ is invertible in } B(X)\}$$

be the *conjugate class*, or the *similarity orbit*, of T.

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Theorem (with Bès, 2003)

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The similarity orbit $\mathcal{S}(T)$ of a hypercyclic operator T contains a path $\{F_t\}$ that is SOT-dense in $B(X)$ and yet the path has a dense G_δ set of common hypercyclic vectors. Furthermore, for any $g \in \mathcal{HC}(T)$, the path of operators may be chosen such that $\{\text{span Orb}(T, g)\} \setminus \{0\} \subseteq \bigcap_{t\in[1,\infty)} \mathcal{HC}(F_t)$.

Corollary 1

Let g be any nonzero vector in X. Then the set

$$\mathcal{A} = \{A \in B(X) : g \in \mathcal{HC}(A)\}$$

is SOT-dense and SOT-connected in $B(X)$. Furthermore,

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Corollary 2

The set of all hypercyclic operators in $B(X)$ is SOT-connected.

Proof of Corollary 1. To prove the first part, it suffices to show there is an operator $T \in B(X)$ for which $g \in \mathcal{HC}(T)$.

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Let T_0 be a hypercyclic operator with $\mathcal{HC}(T_0) \neq X \setminus \{0\}$. Hence, there is an invertible operator L such that $g \in \mathcal{HC}(L^{-1}T_0L)$.

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For the second part, observe that

$$(\text{span}\{g\}) \setminus \{0\} \subseteq \bigcap_{A \in \mathcal{A}} \mathcal{HC}(A).$$

Let $h \notin \text{span}\{g\}$. Again, there is an invertible operator L_0 such that $g \in \mathcal{HC}(L_0^{-1}T_0L_0)$ and $h \notin \mathcal{HC}(L_0^{-1}T_0L_0)$. Thus

$$h \notin \bigcap_{A \in \mathcal{A}} \mathcal{HC}(A). \quad \square$$

Suppose the Banach space X is a Hilbert space H.

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There is a path of hypercyclic operators in $B(H)$ which is SOT-dense in $B(H)$ and every operator along the path has the exact same dense G_δ set $\mathcal{G}$ of hypercyclic vectors.

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Corollary

The set of operators T in $B(H)$ with $\mathcal{G} \subset \mathcal{HC}(T)$ is SOT-connected.

The similarity orbit $\mathcal{S}(T)$ contains a suborbit, called the *unitary orbit* $\mathcal{U}(T) = \{U^{-1}TU : U$ unitary in $B(H)\}$.

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Theorem (2011)

If T is a hypercyclic operator in $B(H)$, and $g \in \mathcal{HC}(T)$, then there exists a path of operators $\{F_t\}$ contained entirely in $\mathcal{U}(T)$ for which $\mathcal{U}(T) \subseteq \overline{\{F_t\}}^{SOT}$ and the dense G_δ set of common hypercyclic vectors contains all nonzero vectors in span Orb(T, g).

Corollary

Let $\{u_i \in H : i \geq 1\}$ be a linearly independent subset of H that spans a dense linear manifold of H, and let T in $B(H)$ be a hypercyclic operator. Then there exists a path of operators $\{G_t\}$ contained in $\mathcal{U}(T)$ for which $\mathcal{U}(T) \subseteq \overline{\{G_t\}}^{SOT}$ and (span$\{u_i : i \geq 1\}) \setminus \{0\} \subseteq \bigcap_{t \in [0,\infty)} \mathcal{HC}(F_t)$.