Taylor’s Theorem

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Taylor Polynomials

Definition: [Taylor Polynomials]

Assume that $f(x)$ is n-times differentiable at $x = a$. The *n-th degree Taylor polynomial for $f(x)$ centered at $x = a$* is the polynomial

$$
\begin{aligned}
T_{n,a}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \\
&\quad \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n
\end{aligned}
$$

Observation: We have seen that for functions $f(x)$ such as $\cos(x)$, $\sin(x)$ and e^x, that

$$T_{n,a}(x) \cong f(x)$$

near $x = a$.

Taylor's Remainder

Definition: [Taylor Remainder]

Assume that $f(x)$ is n times differentiable at $x = a$. Let

$$R_{n,a}(x) = f(x) - T_{n,a}(x).$$

$R_{n,a}(x)$ is called the *n-th degree Taylor remainder function centered at* $x = a$.

Note: The error in using a Taylor Polynomial to approximate $f(x)$ is given by

$$\textbf{Error} = \mid R_{n,a}(x) \mid \ .$$

Central Problem: Given a function $f(x)$ and a point $x = a$, how do we estimate the size of the Taylor Remainder $R_{n,a}(x)$?

Taylor's Theorem

Theorem: [Taylor's Theorem]

Assume that $f(x)$ is $n+1$-times differentiable on an interval I containing $x=a$. Let $x \in I$. Then there exists a point c between x and a such that

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Remarks:

1) Since $T_{0,a}(x) = f(a)$, when $n = 0$

$$f(x) - T_{0,a}(x) = f(x) - f(a) = f'(c)(x-a)$$

which is the Mean Value Theorem.

2) Since $T_{1,a}(x) = L_a^f(x)$,

$$|f(x) - L_a^f(x)| = |R_{1,a}(x)| = \frac{|f''(c)|}{2}(x-a)^2$$

is the error in using the linear approximation.

Taylor's Theorem

Remarks (continued):

3) Taylor's Theorem does not tell us how to find the point c, only that it exists. Therefore, to estimate $R_{n,a}(x)$ we must first estimate how large $|f^{(n+1)}(c)|$ could be without knowing the value of c.

Taylor's Theorem

Example: Use linear approximation to estimate $\sin(.01)$ and show that the error in using this approximation is less than 10^{-4}.

Solution: We know that $f(0) = \sin(0) = 0$ and that $f'(0) = \cos(0) = 1$, so

$$L_0(x) = T_{1,0}(x) = x.$$

Therefore,

$$\sin(.01) \cong L_0(.01) = .01$$

Since $f(x) = \sin(x)$, $f'(x) = \cos(x)$, and $f''(x) = -\sin(x)$, Taylor's Theorem guarantees that there exists some c between 0 and $.01$ such that the error in the linear approximation is given by

$$\begin{aligned} |R_{1,0}(.01)| &= \left| \frac{f''(c)}{2}(.01 - 0)^2 \right| \\ &= \left| \frac{-\sin(c)}{2}(.01)^2 \right| \\ &\leq \frac{(.01)^2}{2} \end{aligned}$$

since $|-\sin(c)| \leq 1$.

Taylor's Theorem

Remark: For $f(x) = \sin(x)$, we saw that

$$L_0(x) = T_{1,0}(x) = x.$$

However, since $f''(0) = -\sin(0) = 0$, we also have

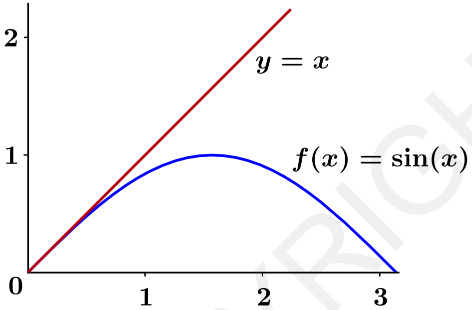
$$L_0(x) = T_{1,0}(x) = T_{2,0}(x)$$

so there exists a c between 0 and $.01$ such that

$$\begin{aligned} |\sin(.01) - .01| &= |R_{2,0}(.01)| \\ &= \left|\frac{f'''(c)}{6}(.01-0)^3\right| \\ &= \left|\frac{-\cos(c)}{6}(.01)^3\right| \\ &< 10^{-6} \end{aligned}$$

since $|-\cos(c)| \leq 1$.

Taylor's Theorem



Remark: It can be shown that

$$\sin(x) \leq x$$

for all $x \geq 0$.

We can use Taylor's Theorem to show this for all $x \in [0, \frac{\pi}{2}]$.

The statement is clearly true for $x = 0$. Let $x \in (0, \frac{\pi}{2}]$. Then by Taylor's Theorem there is a $c \in (0, x)$ with

$$\sin(x) - x = R_{1,0}(x) = \frac{-\sin(c)}{2}x^2 < 0$$

since $\sin(c) > 0$ for any $c \in (0, \frac{\pi}{2})$.