The Taylor Series for Cosine and Sine

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Taylor Series for $\cos(x)$

Example: Find the Taylor series centered at $x = 0$ for $f(x) = \cos(x)$.

$$
\begin{array}{lclclclcr}
f'(x) & = & -\sin(x) & \Longrightarrow & f'(0) & = & -\sin(0) & = & 0 \\
f''(x) & = & -\cos(x) & \Longrightarrow & f''(0) & = & -\cos(0) & = & -1 \\
f'''(x) & = & \sin(x) & \Longrightarrow & f'''(0) & = & \sin(0) & = & 0 \\
f^{(4)}(x) & = & \cos(x) & \Longrightarrow & f^{(4)}(0) & = & \cos(0) & = & 1 \\
f^{(5)}(x) & = & -\sin(x) & \Longrightarrow & f^{(5)}(0) & = & -\sin(0) & = & 0 \\
f^{(6)}(x) & = & -\cos(x) & \Longrightarrow & f^{(6)}(0) & = & -\cos(0) & = & -1 \\
f^{(7)}(x) & = & \sin(x) & \Longrightarrow & f^{(7)}(0) & = & \sin(0) & = & 0 \\
f^{(8)}(x) & = & \cos(x) & \Longrightarrow & f^{(8)}(0) & = & \cos(0) & = & 1
\end{array}
$$

$$\vdots$$

In fact,

$$
\begin{array}{lclclclcr}
f^{(4k)}(x) & = & \cos(x) & \Longrightarrow & f^{(4k)}(0) & = & \cos(0) & = & 1 \\
f^{(4k+1)}(x) & = & -\sin(x) & \Longrightarrow & f^{(4k+1)}(0) & = & -\sin(0) & = & 0 \\
f^{(4k+2)}(x) & = & -\cos(x) & \Longrightarrow & f^{(4k+2)}(0) & = & -\cos(0) & = & -1 \\
f^{(4k+3)}(x) & = & \sin(x) & \Longrightarrow & f^{(4k+3)}(0) & = & \sin(0) & = & 0
\end{array}
$$

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Taylor Series for $\cos(x)$

Example (continued):

Find the Taylor series centered at $x = 0$ for $f(x) = \cos(x)$.

Hence

$$\begin{aligned} \cos(x) &\sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= 1 + \frac{0x}{1!} + \frac{-1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \end{aligned}$$

Note: The series converges for all $x \in \mathbb{R}$.

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Taylor Series for $\sin(x)$

Example: Find the Taylor series centered at $x = 0$ for $g(x) = \sin(x)$.

We have

$$\begin{array}{rclcrclcr} g^{(4k)}(x) & = & \sin(x) & \Longrightarrow & g^{(4k)}(0) & = & \sin(0) & = & 0 \\ g^{(4k+1)}(x) & = & \cos(x) & \Longrightarrow & g^{(4k+1)}(0) & = & \cos(0) & = & 1 \\ g^{(4k+2)}(x) & = & -\sin(x) & \Longrightarrow & g^{(4k+2)}(0) & = & -\sin(0) & = & 0 \\ g^{(4k+3)}(x) & = & -\cos(x) & \Longrightarrow & g^{(4k+3)}(0) & = & -\cos(0) & = & -1 \end{array}$$

Hence

$$\begin{aligned} \sin(x) &\sim x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

Question: Does

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

and

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \; ?$$

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Taylor Series for $\cos(x)$ and $sin(x)$

Key Observation: Suppose that f is a function for which $f^{(n)}(a)$ exists for each n and hence with Taylor series

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Then the k-th partial sum of the Taylor Series is

$$T_{k,a}(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

which is the k-th degree Taylor polynomial for f centered at $x = a$.

Hence

$$f(x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x_0-a)^n = \lim_{k\to\infty} T_{k,a}(x_0)$$

if and only if

$$\lim_{k\to\infty} R_{k,a}(x_0) = 0$$

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