Taylor Polynomials

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Linear Approximation

Recall:

Definition: [Linear Approximation]

If $f(x)$ is differentiable at $x = a$, then

$$L_a^f(x) = f(a) + f'(a)(x - a)$$

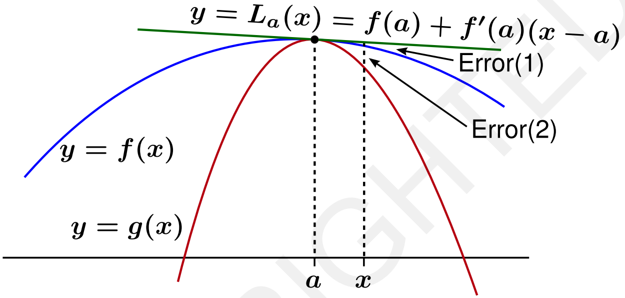
is called the *linear approximation to* $f(x)$ *centered at* $x = a$.

Key Properties:

1. $L_a^f(a) = f(a)$.
2. $(L_a^f)'(a) = f'(a)$.
3. $L_a^f(x)$ is the unique function of the form $y = c_o + c_1(x - a)$ satisfying (1) and (2).
4. If $x \cong a$, then $L_a^f(x) \cong f(x)$.

Note: The graph of $L_a^f(x)$ is the **tangent line** to the graph of $f(x)$ through $(a, f(a))$.

Error in Approximating Functions



Observation : The error in linear approximation

$$|\ f(x) - L_a^f(x)\ |$$

depends on two factors:

1) The distance from x to $a \Rightarrow |\ x - a\ |$.
2) The *curvature* of the graph of $f(x)$ near $x = a \Rightarrow |f''(x)|$ near $x = a$.

Question 1: Can we approximate $f(x)$ better by using a second degree polynomial that also encodes $f''(a)$?

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Question 2: Can we find a polynomial

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2$$

such that

$$\begin{aligned} p(a) &= f(a), \\ p'(a) &= f'(a), \\ p''(a) &= f''(a)? \end{aligned}$$

Solution:

i) $f(a) = p(a) = c_0 + c_1(a-a) + c_2(a-a)^2 = c_0 \Rightarrow c_0 = f(a)$.

ii) $p'(x) = c_1 + 2 \cdot c_2(x-a) \Rightarrow f'(a) = p'(a) =$
$c_1 + 2 \cdot c_2(a-a) = c_1 \Rightarrow c_1 = f'(a)$

iii) $p''(x) = 2 \cdot c_2 \Rightarrow f''(a) = p''(a) = 2 \cdot c_2 \Rightarrow c_2 = \frac{f''(a)}{2}$

Note: This polynomial, denoted by

$$T_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = L_a^f(x) + \frac{f''(a)}{2}(x-a)^2,$$

is called the *second degree Taylor polynomial of $f(x)$ centered at $x = a$.*

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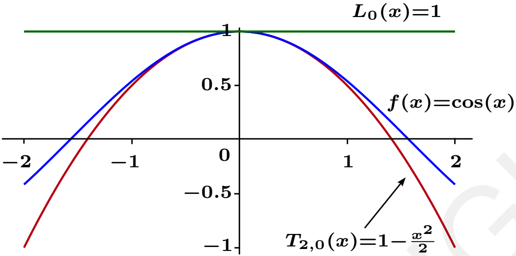

Example :

Let $f(x) = \cos(x)$.

Then,

$$\begin{aligned} f(0) &= \cos(0) = 1, \\ f'(0) &= -\sin(0) = 0, \\ f''(0) &= -\cos(0) = -1. \end{aligned}$$

So

$$L_0(x) = f(0) + f'(0)(x-0) = 1 + 0(x-0) = 1$$

for all x while

$$\begin{aligned} T_{2,0}(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 \\ &= 1 + 0(x-0) + \frac{-1}{2}(x-0)^2 \\ &= 1 - \frac{x^2}{2}. \end{aligned}$$

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Question 3: Can we encode more derivatives?

Answer: If

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3,$$

then

1. $p(a) = f(a)$,
2. $p'(a) = f'(a)$,
3. $p''(a) = f''(a)$, and
4. $p'''(a) = f'''(a)$.

Notation: In this case, we call $p(x)$ the *third degree Taylor polynomial centered at* $x = a$ and denote it by $T_{3,a}(x)$.

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Definition: [Taylor Polynomials]

Assume that $f(x)$ is n-times differentiable at $x = a$. The *n-th degree Taylor polynomial for $f(x)$ centered at $x = a$* is the polynomial

$$\begin{aligned} T_{n,a}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \\ &\quad \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Observation: Using the convention where $0! = 1! = 1$ and $(x-a)^0 = 1$, we have the following:

$$\begin{aligned} T_{0,a}(x) &= \frac{f(a)}{0!}(x-a)^0 = f(a) \\ T_{1,a}(x) &= \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 = L_a^f(x) \\ T_{2,a}(x) &= \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 \end{aligned}$$

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Key Observation: A remarkable property about $T_{n,a}(x)$ is that for any k between 0 and n,

$$T_{n,a}^{(k)}(a) = f^{(k)}(a).$$

That is, $T_{n,a}(x)$ encodes not only the value of $f(x)$ at $x = a$ but all of its first n derivatives as well. Moreover, this is the *only* polynomial of degree n or less that does so.