Introduction to Taylor Series

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Taylor Series

Remark : Given a function f that can be represented by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

centered at $x = a$ with $R > 0$, we have seen that f has derivatives of all orders at $x = a$ and that

$$a_n = \frac{f^{(n)}(a)}{n!}$$

So, in fact,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Question : If we assume that a function f has derivatives of all orders at $a \in \mathbb{R}$, then the above series can certainly be constructed. Does the series represent f?

Taylor Series

Definition: [Taylor Series]

Assume that f has derivatives of all orders at $a \in \mathbb{R}$. The formal series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the *Taylor series* for f centered at $x = a$.

We write

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In the special case where $a = 0$, the series is often referred to as the *Maclaurin series* for f.

Key Fact :
If f is represented by any power series it must be its Taylor series.

Taylor Series

Central Problems: Suppose that f is any function for which $f^{(n)}(a)$ exists for each n.

1) For which values of x does the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

converge?

2) If the series above converges at x_0, is it true that

$$f(x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x_0-a)^n \text{ ?}$$

That is, can f be reconstructed from the data contained in its various derivatives at the single point $x = a$?

Taylor Series

Example: Let $f(x) = e^x$. Then

$$f^{(k)}(x) = e^x \quad \text{and} \quad f^{(k)}(0) = e^0 = 1$$

for all $k = 0, 1, 2, 3, \ldots$

Hence

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and this series converges for all $x \in \mathbb{R}$.

Fact: We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x \in \mathbb{R}$.

Taylor Series

Example: Let

$$g(x) = \begin{cases} \frac{1}{e} & \text{if } x < -1 \\ e^x & \text{if } -1 \le x \le 1 \\ e & \text{if } x > 1 \end{cases}$$

Then

$$g^{(k)}(0) = e^0 = 1$$

for all $k = 0, 1, 2, 3, \ldots$ so that

$$g(x) \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

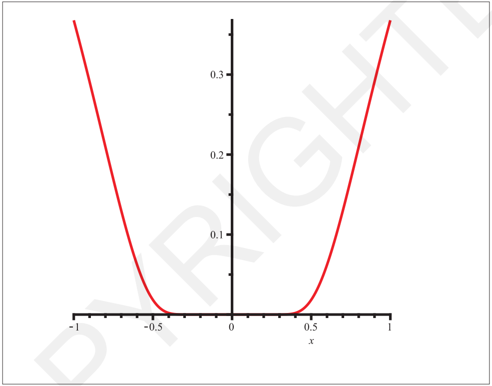
and this series converges for all $x \in \mathbb{R}$.

However

$$g(x) \neq \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$$

if $x = 7$.

A Strange Example:



Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Taylor Series

Strange Example (continued): For

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

we can show that for every $k = 0, 1, 2, \cdots$,

$$f^{(k)}(0) = 0$$

Hence, the Taylor series is

$$f(x) \sim \sum_{n=0}^{\infty} 0 \cdot x^n = 0$$

which converges everywhere but only agrees with the function $f(x)$ at $x = 0$.

Taylor Series

Important Observations:

Suppose that f is any function for which $f^{(n)}(a)$ exists for each n.

1) Many different functions can have the same Taylor series centered at $x = a$ as does f .

2) Even if the Taylor series of f converges at a point x_0, it need not converge to $f(x_0)$.

3) The Taylor series centered at $x = a$ will always converge at a with value $f(a)$.

Taylor Series

Example: Find the Taylor series centered at $x = 0$ for $f(x) = \cos(x)$.

$$
\begin{array}{rcrcrcrcr}
f'(x) & = & \sin(x) & \Longrightarrow & f'(0) & = & -\sin(0) & = & 0 \\
f''(x) & = & -\cos(x) & \Longrightarrow & f''(0) & = & -\cos(0) & = & -1 \\
f'''(x) & = & \sin(x) & \Longrightarrow & f'''(0) & = & \sin(0) & = & 0 \\
f^{(4)}(x) & = & \cos(x) & \Longrightarrow & f^{(4)}(0) & = & \cos(0) & = & 1 \\
f^{(5)}(x) & = & -\sin(x) & \Longrightarrow & f^{(5)}(0) & = & -\sin(0) & = & 0 \\
f^{(6)}(x) & = & -\cos(x) & \Longrightarrow & f^{(6)}(0) & = & -\cos(0) & = & -1 \\
f^{(7)}(x) & = & \sin(x) & \Longrightarrow & f^{(7)}(0) & = & \sin(0) & = & 0 \\
f^{(8)}(x) & = & \cos(x) & \Longrightarrow & f^{(8)}(0) & = & \cos(0) & = & 1 \\
 & & & \vdots & & & & &
\end{array}
$$

In fact,

$$
\begin{array}{rcrcrcrcr}
f^{(4k)}(x) & = & \cos(x) & \Longrightarrow & f^{(4k)}(0) & = & \cos(0) & = & 1 \\
f^{(4k+1)}(x) & = & -\sin(x) & \Longrightarrow & f^{(4k+1)}(0) & = & -\sin(0) & = & 0 \\
f^{(4k+2)}(x) & = & -\cos(x) & \Longrightarrow & f^{(4k+2)}(0) & = & -\cos(0) & = & -1 \\
f^{(4k+3)}(x) & = & \sin(x) & \Longrightarrow & f^{(4k+3)}(0) & = & \sin(0) & = & 0
\end{array}
$$

Taylor Series

Example (continued):

Find the Taylor series centered at $x = 0$ for $f(x) = \cos(x)$.

Hence

$$
\begin{aligned}
\cos(x) &\sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= 1 + \frac{0x}{1!} + \frac{-1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}
\end{aligned}
$$

Note: The series converges for all $x \in \mathbb{R}$.

Taylor Series

Example: Find the Taylor series centered at $x = 0$ for $g(x) = \sin(x)$.

We have

$$\begin{array}{lclclclcl} g^{(4k)}(x) & = & \sin(x) & \Longrightarrow & g^{(4k)}(0) & = & \sin(0) & = & 0 \\ g^{(4k+1)}(x) & = & \cos(x) & \Longrightarrow & g^{(4k+1)}(0) & = & \cos(0) & = & 1 \\ g^{(4k+2)}(x) & = & -\sin(x) & \Longrightarrow & g^{(4k+2)}(0) & = & -\sin(0) & = & 0 \\ g^{(4k+3)}(x) & = & -\cos(x) & \Longrightarrow & g^{(4k+3)}(0) & = & -\cos(0) & = & -1 \end{array}$$

Hence

$$\begin{aligned} \sin(x) &\sim x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

Question: Does

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

and

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \ ?$$

Taylor Series

Key Observation: Suppose that f is a function for which $f^{(n)}(a)$ exists for each n and hence with Taylor series

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Then the k-th partial sum of the Taylor Series is

$$T_{k,a}(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

which is the k-th degree Taylor polynomial for f centered at $x = a$.

Hence

$$f(x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x_0-a)^n = \lim_{k\to\infty} T_{k,a}(x_0)$$

if and only if

$$\lim_{k\to\infty} R_{k,a}(x_0) = 0$$