Introduction to Power Series

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Example

Problem: For which values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

Solution: Observe that if $b_n = \mid \frac{x^n}{n} \mid$ then

$$\begin{aligned} \lim_{n\to\infty} \frac{b_{n+1}}{b_n} &= \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{n+1}}{\frac{|x|^n}{n}} \\ &= \lim_{n\to\infty} \mid x \mid \frac{n}{n+1} \\ &= \mid x \mid \end{aligned}$$

The Ratio Test shows that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges absolutely if $\mid x \mid < 1$ and the series diverges if $\mid x \mid > 1$.

If $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

If $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges.

Power Series

Definition: [Power Series]

A *power series centered at* $x = a$ is a formal series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

where x is viewed as a variable.

The value a_n is called the *coefficient* of the term $(x-a)^n$.

Central Questions:

1) If $I = \{x_0 \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n (x_0 - a)^n \text{ converges}\}$, then what can we say about I?
2) If we define a function f on I by $f(x_0) = \sum_{n=0}^{\infty} a_n (x_0 - a)^n$ for all $x_0 \in I$, then what can we say about $f(x)$?

Power Series

Observations:

1) Our convention is that $0^0 = 1$, so that if $x = a$, the series becomes

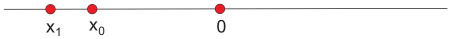
$$\sum_{n=0}^{\infty} a_n (a-a)^n = a_0 + 0 + 0 + 0 + \cdots = a_0$$

2) We can let $u = x - a$ to get that $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges at $x = x_0$ if and only if $\sum_{n=0}^{\infty} a_n u^n$ converges at $u = x_0 - a$.

Consequence: We need only focus on series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

Interval of Convergence



$$|x_0| < |x_1| \implies \frac{|x_0|}{|x_1|} < 1$$

Note: The series $\sum_{n=0}^{\infty} a_n x^n$ always converges at $x = 0$. Assume that $\sum_{n=0}^{\infty} a_n x_1^n$ converges where $x_1 \neq 0$.

Then $\lim_{n\to\infty} | a_n x_1^n | = 0$ and there exists an $N_0 \in \mathbb{N}$ so that if $n \geq N_0$ we have $| a_n x_1^n | \leq 1$.

Let $| x_0 | < | x_1 |$. Then if $n \geq N_0$

$$| a_n x_0^n | = | a_n x_1^n | \left| \frac{x_0}{x_1} \right|^n \leq \left| \frac{x_0}{x_1} \right|^n$$

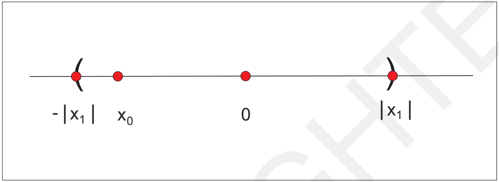
Since $\left| \frac{x_0}{x_1} \right| < 1$, the series

$$\sum_{n=N_0}^{\infty} \left| \frac{x_0}{x_1} \right|^n$$

converges. By the Comparison Test we have that

$$\sum_{n=0}^{\infty} a_n x_0^n \quad \text{converges absolutely.}$$

Interval of Convergence



Key Observations: If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x_1 \neq 0$ and if $|x_0| < |x_1|$, then the series converges at x_0 as well.

Let

$$I = \{x_0 \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n x_0^n \text{ converge}\}.$$

Then

$$(-|x_1|, |x_1|) \subset I$$

$\Rightarrow I$ is an interval centered around $x = 0$.

Note: The same analysis works for power series of the form $\sum_{n=0}^{\infty} a_n (x-a)^n$.

Interval and Radius of Convergence

Definition: [Interval and Radius of Convergence]

Given a power series of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$, the set

$$I = \{x_0 \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x_0-a)^n \text{ converges}\}$$

is an interval centered at $x = a$ which we call *the interval of convergence* for the power series.

Let

$$R = \begin{cases} lub(\{|x_0 - a| \mid x_0 \in I\}) & \text{if } I \text{ is bounded,} \\ \infty & \text{if } I \text{ is not bounded.} \end{cases}$$

Then R is called the *radius of convergence* of the power series.

Interval of Convergence

Theorem: [Fundamental Convergence Theorem for Power Series]

Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ centered at $x = a$, let R be the radius of convergence.

1. If $R = 0$, then $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $x = a$, but it diverges for all other values of x.
2. If $0 < R < \infty$, then the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for every $x \in (a-R, a+R)$ and diverges if $| x - a | > R$.
3. If $R = \infty$, then the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for every $x \in \mathbb{R}$.

Interval of Convergence

Remark: If $0 < R < \infty$, then there are four possibilities for the interval of convergence I.

1) $I = (a - R, a + R)$ Example: $\sum_{n=0}^{\infty} x^n \Rightarrow I = (-1, 1)$.

2) $I = [a - R, a + R)$ Example: $\sum_{n=1}^{\infty} \frac{x^n}{n} \Rightarrow I = [-1, 1)$.

3) $I = (a - R, a + R]$ Example: $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \Rightarrow I = (-1, 1]$.

4) $I = [a - R, a + R]$ Example: $\sum_{n=1}^{\infty} \frac{x^n}{n^2} \Rightarrow I = [-1, 1]$.

Key Note: Once R is determined, you need to test the endpoints separately.