Integration of Power Series

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Formal Antiderivative of a Power Series

Definition: [Formal Antiderivative of a Power Series]

Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, we define *the formal antiderivative* to be the power series

$$\sum_{n=0}^{\infty} \int a_n(x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

where C is an arbitrary constant.

Formal Antiderivative of a Power Series

Fundamental Problem: Suppose that the power series

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

has radius of convergence $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

be the function that is represented by this power series on the interval $(a-R, a+R)$. Are the formal anitderivatives

$$C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}$$

true antiderivatives of the function f?

Formal Antiderivative of a Power Series

Key Observations:

1. The series
$$\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^n$$
is obtained from $\sum_{n=0}^{\infty} a_n(x-a)^n$ by dividing each coefficient a_n by $q(n) = n+1$ so they have the same radius of convergence R.
2. The formal antiderivative
$$C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$
will have radius of convergence R.
3. If
$$F(x) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$
on $(a-R, a+R)$, then its formal derivative is
$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

Term-by-Term Integration of a Power Series

Theorem: [Term-by-Term Integration of a Power Series]

Assume that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for every $x \in (a-R, a+R)$. Then the series

$$\sum_{n=0}^{\infty} \int a_n(x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

also has radius of convergence R and if

$$F(x) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

then $F'(x) = f(x)$.
Furthermore, if $[c, b] \subset (a-R, a+R)$, then

$$\int_c^b f(x)\, dx = \int_c^b \sum_{n=0}^{\infty} a_n(x-a)^n \, dx = \sum_{n=0}^{\infty} \int_c^b a_n(x-a)^n \, dx$$

Term-by-Term Integration of a Power Series

Important Note: It may seem perfectly natural that we are also able to integrate term-by-term the functions that are represented by a power series.

In general, if

$$F(x) = \sum_{n=1}^{\infty} f_n(x)$$

for each $x \in [a, b]$, then we might hope that

$$\int_a^b F(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

Fact: If we do not make any additional assumptions about the nature of the functions f_n or about how the series converges, then it is possible that the function F need not even be integrable on $[a, b]$ even if all of the f_n's are.

Term-by-Term Integration of a Power Series

Example: Find a power series representation for $\ln(1+x)$.

Solution: Note that

$$\frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x}$$

Key Observation: For any $u \in (-1, 1)$,

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

If $u = -x$ we get

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

for any $x \in (-1, 1)$.

Term-by-Term Integration of a Power Series

Solution (continued): Therefore, there is a constant C such that

$$\begin{aligned} \ln(1+x) &= C + \sum_{n=0}^{\infty} \int (-1)^n x^n \, dx \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \end{aligned}$$

for all $x \in (-1, 1)$.

When $x = 0$ we get

$$\begin{aligned} 0 &= \ln(1+0) \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} 0^{n+1} \\ &= C \end{aligned}$$

Therefore

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

for all $x \in (-1, 1)$.

Term-by-Term Integration of a Power Series

Key Note: The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

has radius of convergence $R = 1$. However, if $x = 1$ the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} 1^{n+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

which is exactly the Alternating Series. Therefore, the series also converges at $x = 1$.

By Abel's Theorem, the equation

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

is actually valid on (-1,1] and

$$\ln(2) = \ln(1+1) = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$