Functions Represented by Power Series

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Functions Represented by Power Series

Example: Recall that the geometric series

$$\sum_{n=0}^{\infty} x^n$$

converges for each x with $| x |< 1$.

Moreover, if $| x |< 1$, then we also know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Note: This means that the series provides us with a means to *represent* the function $f(x) = \frac{1}{1-x}$ on the interval $(-1, 1)$.

Functions Represented by Power Series

Definition: [Functions Represented by a Power Series]

Let $\sum_{n=0}^{\infty} a_n(x-a)^n$ be a power series with radius of convergence $R > 0$. Let I be the interval of convergence for $\sum_{n=0}^{\infty} a_n(x-a)^n$.

Let f be the function defined on the interval I by the formula

$$f(x_0) = \sum_{n=0}^{\infty} a_n(x_0-a)^n \quad (*)$$

for each $x_0 \in I$.

We say that the function f *is represented by the power series* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *on* I.

Functions Represented by Power Series

Example: Find a power series representation for the function

$$f(x) = \frac{x}{1 - x^2}$$

on the interval $(-1, 1)$.

Solution: We know for any $-1 < u < 1$ that

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

If $-1 < x < 1$, and if $u = x^2$, then $u \in (-1, 1)$, so

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$$

Hence

$$\begin{aligned}
\frac{x}{1-x^2} &= x \cdot \sum_{n=0}^{\infty} x^{2n} \\
&= x \cdot \lim_{k \to \infty} \sum_{n=0}^{k} x^{2n} \\
&= \lim_{k \to \infty} \sum_{n=0}^{k} x^{2n+1} = \sum_{n=0}^{\infty} x^{2n+1}
\end{aligned}$$

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Example: Find the function represented by the power series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \cdots$$

on the interval $(-1, 1)$.

Observation: If $x \in (-1, 1)$, then

$$\begin{aligned} 1 + 2x + x^2 + 2x^3 + x^4 + \cdots &= (1 + x + x^2 + x^3 + \cdots) \\ &\quad +(x + x^3 + x^5 + \cdots) \\ &= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{2n+1} \\ &= \frac{1}{1-x} + \frac{x}{1-x^2} \\ &= \frac{1+x}{1-x^2} + \frac{x}{1-x^2} \\ &= \frac{1+2x}{1-x^2} \end{aligned}$$

for all $x \in (-1, 1)$.

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Question: Assume that the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$ and interval of convergence I. What are the properties of the function

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \; ?$$

1. Is f continuous on I?
2. Is f differentiable on $(-R, R)$? And if so what is its derivative?

Functions Represented by Power Series

Theorem: [Abel's Theorem: Continuity of Power Series]

Assume that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has interval of convergence I. Let

$$f(x_0) = \sum_{n=0}^{\infty} a_n(x_0 - a)^n$$

for each $x_0 \in I$. Then f is continuous on I.