Differentiation of Power Series

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Functions Represented by Power Series

Question: Assume that the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$ and interval of convergence I. What are the properties of the function

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n?$$

1. Is f continuous on I?

Theorem: [Abel's Theorem: Continuity of Power Series]

Assume that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has interval of convergence I. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for each $x \in I$. Then f is continuous on I.

2. Is f differentiable on $(a-R, a+R)$?

Differentiation of Power Series

Strategy: If we have a function

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

that is represented by a power series with radius of convergence $R > 0$, we could try to *differentiate f by differentiating the series one term at a time.*

Since $\frac{d}{dx}(a_n(x-a)^n) = na_n(x-a)^{n-1}$, we get:

Definition: [Formal Derivative of a Power Series]

Given a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$, the *formal derivative* is the series

$$\sum_{n=1}^{\infty} na_n (x-a)^{n-1}$$

Differentiation of Power Series

Two Fundamental Problems:

Problem 1: For which values of x does the formal power series

$$\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

converge? In particular, does this series converge for the same values as the original series $\sum_{n=0}^{\infty} a_n (x-a)^n$?

Problem 2: If both of the series $\sum_{n=0}^{\infty} a_n (x-a)^n$ and $\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$ converge at the same x, must it be the case that

$$f\,'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}?$$

Differentiation of Power Series

Problem 1: For which values of x does the formal power series

$$\sum_{n=1}^{\infty} na_n(x-a)^{n-1}$$

converge? In particular, does this series converge for the same values as the original series $\sum_{n=0}^{\infty} a_n(x-a)^n$?

Observation: The series $\sum_{n=0}^{\infty} a_n(x-a)^n$ and the series $\sum_{n=0}^{\infty} na_n(x-a)^n$ have the same radius of convergence.

We can show that the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ and its formal derivative $\sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ also have the same radius of convergence, though the interval of convergence may be different. Therefore,

$$g(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$$

is defined for all $x \in (a-R, a+R)$. Is $g(x) = f'(x)$?

Differentiation of Power Series

Theorem: [Differentiation of Power Series]

Assume that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in (a-R, a+R)$. Then f is differentiable on $(a-R, a+R)$ and for each $x \in (a-R, a+R)$,

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x-a)^{n-1}$$

Differentiation of Power Series

Example: If $|x| < 1$, then let

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Differentiating term-by-term, we get

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

Question: Evaluate

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$$

Observation: This series is obtained from $\sum_{n=1}^{\infty} n x^{n-1}$ by letting $x = \frac{1}{2}$.

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} &= f'\left(\frac{1}{2}\right) \\ &= \frac{1}{(1-\frac{1}{2})^2} \\ &= 4 \end{aligned}$$

Differentiation of Power Series

Example: For any $x \in \mathbb{R}$ let

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Term-by-term differentiation gives

$$\begin{aligned} g'(x) &= \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= g(x) \end{aligned}$$

Hence

$$g(x) = Ce^x$$

for some constant C. However, $C = g(0) = 1$, so

$$g(x) = e^x$$

Differentiation of Power Series

Example: Find a power series representation for the function

$$f(x) = e^{-x^2}$$

We have that for any $u \in \mathbb{R}$ that

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad (*)$$

Let $u = -x^2$ and substitute for u in the expression $(*)$ to get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

for all $x \in \mathbb{R}$.

Note: It may look like

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots$$

is not a power series since there are no terms involving x^n when n is odd. But in fact, it really is a power series where the coefficients are of the form $a_{2k-1} = 0$ and $a_{2k} = (-1)^k \frac{1}{(k)!}$ for each $k = 0, 1, 2, 3, 4, \ldots$

A Strange Function

Question: Why are power series so special?

Example: Let

$$f(x) = \sum_{n=0}^{\infty} (\frac{3}{4})^n \sin(9^n x)$$

for all $x \in \mathbb{R}$.

Fact: The function f is continuous on $\mathbb{R}$ but it is not differentiable at a single point.