Convergence of Taylor Series

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Convergence of Taylor Series

Recall: We have seen that $\cos(x)$ and $\sin(x)$ have Taylor series centered at $x = 0$ as follows:

$$\cos(x) \sim \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

and

$$\sin(x) \sim \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Question: Does

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

and

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} ?$$

Taylor Series

Key Observation: Suppose that f is a function for which $f^{(n)}(a)$ exists for each n and hence with Taylor series

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Then the k-th partial sum of the Taylor Series is

$$T_{k,a}(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

which is the k-th degree Taylor polynomial for f centered at $x = a$.

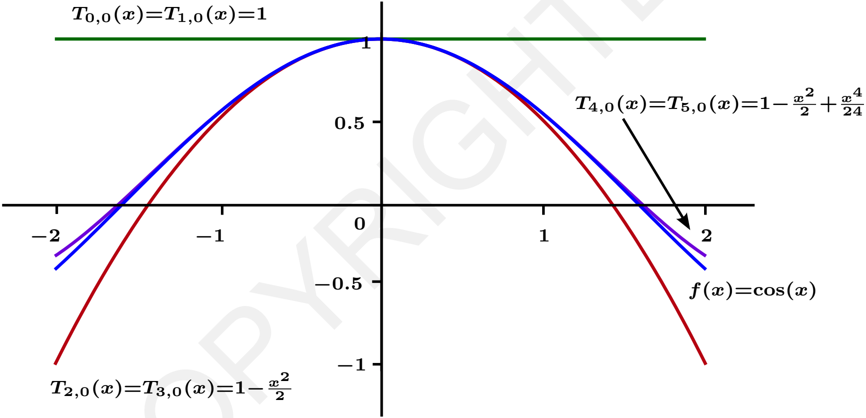
Hence

$$f(x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x_0-a)^n = \lim_{k\to\infty} T_{k,a}(x_0)$$

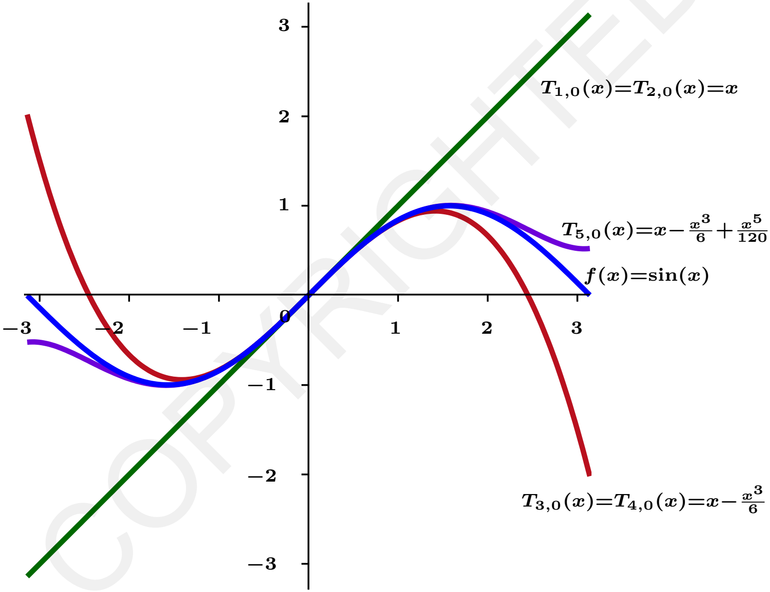
if and only if

$$\lim_{k\to\infty} R_{k,a}(x_0) = 0$$

Convergence of Taylor Series



Convergence of Taylor Series



Taylor Series

Key Tool to Use:

Recall that the Ratio Test showed that for any $x_0 \in \mathbb{R}$, we have

$$\lim_{k \to \infty} \frac{M \mid x_0 \mid^k}{k!} = 0$$

Convergence of Taylor Series:

Examples: Show that for each $x \in \mathbb{R}$,

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Note: If $f(x) = \cos(x)$, then

$$\begin{aligned} f^{(4k)}(x) &= \cos(x) \\ f^{(4k+1)}(x) &= -\sin(x) \\ f^{(4k+2)}(x) &= -\cos(x) \\ f^{(4k+3)}(x) &= \sin(x) \end{aligned}$$

Therefore, for each $x_0 \in \mathbb{R}$ and each $k \in \mathbb{N} \cup \{0\}$,

$$|f^{(k)}(x_0)| \leq 1$$

By Taylor's Theorem, if $x_0 \in \mathbb{R}$,

$$|R_{k,0}(x_0)| = \frac{|f^{(k+1)}(c_k)|}{(k+1)!} |x_0|^{k+1} \leq \frac{|x_0|^{k+1}}{(k+1)!}$$

Hence by the Squeeze Theorem

$$\lim_{k \to \infty} |R_{k,0}(x_0)| = 0 \quad \text{and} \quad \cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Convergence of Taylor Series

Theorem: [Convergence Theorem for Taylor Series]

Assume that f has derivatives of all orders on an interval I containing $x = a$. Assume also that there exists an M such that

$$| f^{(k)}(x) | \leq M$$

for all k and for all $x \in I$. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $x \in I$.

Proof: We know that the Taylor series converges at $x = a$ and that

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(a-a)^n = f(a) + 0 + 0 + 0 + \cdots = f(a)$$

Convergence of Taylor Series

Proof (continued):

Choose $x_0 \in I$ with $x_0 \neq a$. Let $k \in \mathbb{N} \cup \{0\}$. Then Taylor's Theorem tells us that there exists a c_k between a and x_0 so that

$$|f(x_0) - T_{k,a}(x_0)| = \frac{|f^{(k+1)}(c_k)|}{(k+1)!}|x_0 - a|^{k+1}$$

But since

$$| f^{(k+1)}(c_k) | \leq M$$

we have that

$$0 \leq |f(x_0) - T_{k,a}(x_0)| \leq M \cdot \frac{|x_0 - a|^{k+1}}{(k+1)!}$$

Since

$$\lim_{k \to \infty} M \cdot \frac{|x_0 - a|^{k+1}}{(k+1)!} = 0$$

the Squeeze Theorem shows that

$$\lim_{k \to \infty} |f(x_0) - T_{k,a}(x_0)| = 0$$

and hence that

$$f(x_0) = \lim_{k \to \infty} T_{k,a}(x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x_0 - a)^n$$

Convergence of Taylor Series

Example: If $f(x) = \sin(x)$, then

$$| f^{(k)}(x) | \leq 1$$

for all $x \in \mathbb{R}$ and $k = 0, 1, 2, \ldots$.

Hence

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

for all $x \in \mathbb{R}$.

Convergence of Taylor Series

Example: Let $f(x) = e^x$ and $I = [-M, M]$, $M > 0$.
If $x \in [-M, M]$, then

$$| f^{(k)}(x) | = e^x \leq e^M$$

Hence

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x \in [-M, M]$.

Since the above is true for all $M > 0$,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x \in \mathbb{R}$.

Convergence of Taylor Series

Important Remark: The Taylor series can fail to converge back to $f(x_0)$ if the derivatives $f^{(k)}(x_0)$ grow very rapidly as $k \to \infty$.