Building Power Series

Created by

Barbara Forrest and Brian Forrest

Functions Represented by Power Series

Definition: [Functions Represented by a Power Series]

Let $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ be a power series with radius of convergence $R > 0$. Let I be the interval of convergence for $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$.

Let f be the function defined on the interval I by the formula

$$f(x_0) = \sum_{n=0}^{\infty} a_n(x_0-a)^n \quad (*)$$

for each $x_0 \in I$.

We say that the function f *is represented by the power series* $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ *on* I.

Functions Represented by Power Series

Question: Suppose that f and g are represented by power series centered at $x = a$ of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$$

with intervals of convergence I_f and I_g respectively.

Can this information be used to build a power series representation for $f + g$?

Observation: If $x_0 \in I_f \cap I_g$, then

$$\begin{aligned}
(f+g)(x_0) &= f(x_0) + g(x_0) \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} a_n (x_0 - a)^n + \lim_{k\to\infty} \sum_{n=0}^{k} b_n (x_0 - a)^n \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} (a_n + b_n)(x_0 - a)^n \\
&= \sum_{n=0}^{\infty} (a_n + b_n)(x_0 - a)^n
\end{aligned}$$

Sums of Power Series

Theorem: [Addition of Power Series]

Assume that f and g are represented by power series centered at $x = a$ with

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

and

$$g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n,$$

respectively.
Assume also that the radii of convergence of these series are R_f and R_g with intervals of convergence I_f and I_g. Then if $x \in I_f \cap I_g$,

$$(f+g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-a)^n.$$

Moreover, if $R_f \neq R_g$, then the radius of convergence of the power series representing $f+g$ is $R = \min\{R_f, R_g\}$ and the interval of convergence is $I = I_f \cap I_g$.

If $R_f = R_g$, then $R \geq R_f$.

Multiplication by $(x-a)^m$

Remark: Assume that $h(x) = (x-a)^m f(x)$ where $m \in \mathbb{N}$. We might guess that h would be represented by the following power series centered at $x = a$:

$$h(x) = (x-a)^m \sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} a_n (x-a)^{n+m}.$$

Observation: If $x_0 \in I_f$, then

$$\begin{aligned} h(x_0) &= (x_0-a)^m f(x_0) \\ &= (x_0-a)^m \lim_{k\to\infty} \sum_{n=0}^{k} a_n (x_0-a)^n \\ &= \lim_{k\to\infty} \sum_{n=0}^{k} a_n (x_0-a)^{n+m} \\ &= \sum_{n=0}^{\infty} a_n (x_0-a)^{n+m} \end{aligned}$$

Multiplication by $(x-a)^m$

Theorem: [Multiplication of Power Series by $(x-a)^m$]

Assume that f is represented by a power series centered at $x=a$ as

$$f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$$

with radius of convergence R_f and interval of convergence I_f. Assume that $h(x)=(x-a)^m f(x)$ where $m\in\mathbb{N}$. Then h can also be represented by a power series centered at $x=a$ with

$$h(x)=\sum_{n=0}^{\infty} a_n(x-a)^{n+m}$$

Moreover, the series that represents h has the same radius of convergence and the same interval of convergence as the series that represents f.

Multiplication by $(x-a)^m$

Example: Find a power series representation centered at $x=0$ for

$$h(x) = \frac{x}{1-x}.$$

Note: We have that

$$h(x) = x \cdot \frac{1}{1-x}$$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for all $x \in (-1,1)$.

Hence

$$h(x) = \frac{x}{1-x} = x \cdot \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$$

for all $x \in (-1,1)$.

Composition with $c \cdot x^m$

Question: Assume that f has a power series representation

$$f(u) = \sum_{n=0}^{\infty} a_n u^n$$

centered at $u = 0$ with interval of convergence I_f.

Can we find a power series representation for

$$h(x) = f(c \cdot x^m)$$

centered at $x = 0$ where c is some non-zero constant?

Observation: If we choose x_0 so that $c \cdot x_0^m \in I_f$, then substituting $c \cdot x_0^m$ for u gives us

$$h(x_0) = f(c \cdot x_0^m) = \sum_{n=0}^{\infty} a_n (c \cdot x_0^m)^n = \sum_{n=0}^{\infty} (a_n \cdot c^n) x_0^{mn}.$$

Composition with $c \cdot x^m$

Theorem: [Composition with $c \cdot x^m$]

Assume that f has a power series representation

$$f(u) = \sum_{n=0}^{\infty} a_n u^n$$

centered at $u = 0$ with radius of convergence R_f and interval of convergence I_f. Let $h(x) = f(c \cdot x^m)$ where c is a non-zero constant. Then h has a power series representation centered at $x = 0$ of the form

$$h(x) = f(c \cdot x^m) = \sum_{n=0}^{\infty} (a_n \cdot c^n) x^{mn}$$

The interval of convergence is

$$I_h = \{x \in \mathbb{R} \mid c \cdot x^m \in I_f\}$$

and the radius of convergence is $R_h = \sqrt[m]{\frac{R_f}{|c|}}$ if $R_f < \infty$ and $R_h = \infty$ otherwise.

Example

Question: Find a power series representation for $f(x) = \frac{x}{1-2x^2}$ centered at $x = 0$.

Solution: We know that

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

for $u \in (-1, 1)$. Then

$$\frac{1}{1-2x^2} = \sum_{n=0}^{\infty} (2x^2)^n = \sum_{n=0}^{\infty} 2^n x^{2n}$$

provided that

$$2x^2 \in (-1, 1) \Rightarrow x^2 \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Therefore,

$$\frac{x}{1-2x^2} = x \cdot \sum_{n=0}^{\infty} 2^n x^{2n} = \sum_{n=0}^{\infty} 2^n x^{2n+1}$$

if and only if $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.