Binomial Series

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Binomial Theorem

Theorem: [Binomial Theorem]

Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for each $x \in \mathbb{R}$ we have that

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In particular, when $a = 1$ we have

$$(1+x)^n = 1 + \sum_{k=1}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} x^k$$

Binomial Theorem

Observation: Consider the expression

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

If $k > n$, then one of the terms in the expression

$$n(n-1)(n-2)\cdots(n-k+1)$$

will be 0 and hence we would have

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = 0$$

Consequently, we also have

$$(1+x)^n = 1 + \sum_{k=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} x^k$$

Hence the polynomial function $(1+x)^n$ is actually represented by the power series

$$1 + \sum_{k=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} x^k$$

Binomial Series

Question: Suppose that $\alpha \in \mathbb{R}$. Is there an analog of the Binomial Theorem for the function

$$(1+x)^{\alpha} \quad ?$$

Definition: [Generalized Binomial Coefficients and Binomial Series]
Let $\alpha \in \mathbb{R}$ and let $k \in \{0, 1, 2, 3, \ldots\}$. Then we define the generalized binomial coefficient α *choose* k by

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

if $k \neq 0$ and

$$\binom{\alpha}{0} = 1$$

We also define the generalized binomial series for α to be the power series

$$1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

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Remark: If $b_k = \left| \binom{\alpha}{k} \right|$. Then for $k \geq 1$

$$\frac{b_{k+1}}{b_k} = \frac{|\alpha - k|}{k+1}$$

Hence

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \lim_{k\to\infty} \frac{|\alpha - k|}{k+1} = 1$$

Therefore, the radius of convergence for the binomial series is 1.
In particular, the series converges absolutely on $(-1, 1)$.

Remark: Is

$$(1+x)^{\alpha} = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \ ?$$

Binomial Series

Observation 1:

If $k \geq 1$

$$\begin{aligned}
\binom{\alpha}{k+1}(k+1) + \binom{\alpha}{k}k &= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}(\alpha-k) \\
&\quad + \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}(k) \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}(\alpha) \\
&= \alpha\binom{\alpha}{k}
\end{aligned}$$

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Observation 2: Next we let

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

for each $x \in (-1, 1)$. We claim that

$$f'(x) + xf'(x) = \alpha f(x)$$

for each $x \in (-1, 1)$. Term-by-term differentiation gives us that

$$\begin{aligned} f'(x) + xf'(x) &= \sum_{k=1}^{\infty} \binom{\alpha}{k} kx^{k-1} + \sum_{k=1}^{\infty} \binom{\alpha}{k} kx^k \\ &= \binom{\alpha}{1} + \sum_{k=2}^{\infty} \binom{\alpha}{k} kx^{k-1} + \sum_{k=1}^{\infty} \binom{\alpha}{k} kx^k \\ &= \alpha + \sum_{k=1}^{\infty} \binom{\alpha}{k+1} (k+1)x^k + \sum_{k=1}^{\infty} \binom{\alpha}{k} kx^k \\ &= \alpha + \sum_{k=1}^{\infty} \left(\binom{\alpha}{k+1} (k+1) + \binom{\alpha}{k} k \right) x^k \end{aligned}$$

Binomial Series

Observation 2 (continued):

If $k \geq 1$ we have

$$\binom{\alpha}{k+1}(k+1) + \binom{\alpha}{k}k = \alpha\binom{\alpha}{k}$$

It follows that

$$\begin{aligned} f'(x) + xf'(x) &= \alpha + \alpha \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k \\ &= \alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \\ &= \alpha f(x) \end{aligned}$$

as claimed.

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Observation 3: If

$$g(x) = \frac{f(x)}{(1+x)^\alpha},$$

then g is differentiable on $(-1, 1)$ and since $\alpha f(x) = (1+x)f'(x)$,

$$\begin{aligned} g'(x) &= \frac{f'(x)(1+x)^\alpha - \alpha f(x)(1+x)^{\alpha-1}}{(1+x)^{2\alpha}} \\ &= \frac{f'(x)(1+x)^\alpha - (1+x)f'(x)(1+x)^{\alpha-1}}{(1+x)^{2\alpha}} \\ &= \frac{f'(x)(1+x)^\alpha - f'(x)(1+x)^\alpha}{(1+x)^{2\alpha}} \\ &= 0 \end{aligned}$$

Since $g'(x) = 0$ for all $x \in (-1, 1)$, $g(x)$ is constant on this interval. However,

$$g(0) = f(0) = 1.$$

Therefore, $g(x) = 1$ for all $x \in (-1, 1)$. It follows that

$$f(x) = (1+x)^\alpha$$

Generalized Binomial Theorem

Theorem: [Generalized Binomial Theorem]

Let $\alpha \in \mathbb{R}$. Then for each $x \in (-1, 1)$ we have that

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

Binomial Series

Example: Use the Generalized Binomial Theorem to find a power series representation for $(1+x)^{-2}$.

Note: By the Generalized Binomial Theorem we have

$$(1+x)^{-2} = \sum_{k=0}^{\infty} \binom{-2}{k} x^k$$

We have that for $k \geq 1$

$$\binom{-2}{k} = \frac{(-2)(-2-1)\cdots(-2-k+1)}{k!} = (-1)^k(k+1)$$

and

$$\binom{-2}{0} = 1 = (-1)^0(0+1)$$

Therefore,

$$\begin{aligned}(1+x)^{-2} &= \sum_{k=0}^{\infty} (-1)^k (k+1) x^k \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} k x^{k-1}\end{aligned}$$