The Root Test

Created by

Barbara Forrest and Brian Forrest

Ratio Test:

Theorem: [Ratio Test]

Given a series $\sum_{n=0}^{\infty} a_n$, assume that

$$\lim_{n\to\infty} \mid \frac{a_{n+1}}{a_n} \mid = L$$

where $L \in \mathbb{R}$ or $L = \infty$.

1. If $0 \leq L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.
2. If $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.
3. If $L = 1$, then no conclusion is possible.

Root Test:

Note: The following is a related to the Ratio Test with a similar proof:

Theorem: [Root Test]

Given a series $\sum_{n=0}^{\infty} a_n$, assume that

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$$

where $L \in \mathbb{R}$ or $L = \infty$.

1. If $0 \leq L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.
2. If $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.
3. If $L = 1$, then no conclusion is possible.

Example:

Example: Determine if the series $\sum_{n=1}^{\infty} \left(\frac{3n^2+4}{4n^2+5n}\right)^n$ converges or diverges.

Solution: Let $a_n = \left(\frac{3n^2+4}{4n^2+5n}\right)^n$. Then

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{3n^2+4}{4n^2+5n}\right)^n} \\ &= \lim_{n\to\infty} \frac{3n^2+4}{4n^2+5n} \\ &= \frac{3}{4} \end{aligned}$$

Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{3}{4} < 1$ we can use the Root Test to show that

$$\sum_{n=1}^{\infty} \left(\frac{3n^2+4}{4n^2+5n}\right)^n$$

converges.

Example:

Example: Determine if the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ converges or diverges.

Solution: Let $a_n = \left(1+\frac{1}{n}\right)^{n^2}$. Then

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} \\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \\ &= e\end{aligned}$$

Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = e > 1$ we can use the Root Test to show that

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$$

diverges.