The Ratio Test

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Geometric Series

Recall:

Theorem: [Geometric Series Test]

A geometric series $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$.

Moreover, if $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Example:

Example: Does the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge or diverge?

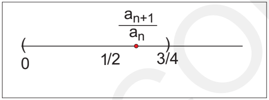
We know $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, but $\frac{1}{2^n} \leq \frac{n}{2^n}$ so the Comparison Test fails even though $n \ll 2^n$.

Key Observation: If $a_n = \frac{n}{2^n}$, then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

We can find an N_0 such that if $n \geq N_0$, then

$$\frac{a_{n+1}}{a_n} < \frac{3}{4} \Rightarrow a_{n+1} < \frac{3}{4} a_n.$$



Example:

Example: (Cont'd:) Therefore,

$$
\begin{aligned}
a_{N_0+1} &< \frac{3}{4}a_{N_0} \\
a_{N_0+2} &< \frac{3}{4}a_{N_0+1} < (\frac{3}{4})^2 a_{N_0} \\
a_{N_0+3} &< \frac{3}{4}a_{N_0+2} < (\frac{3}{4})^3 a_{N_0} \\
&\vdots \\
a_{N_0+k} &< \frac{3}{4}a_{N_0+(k-1)} < (\frac{3}{4})^k a_{N_0}
\end{aligned}
$$

So

$$
\sum_{n=N_0}^{N_0+k} a_n \leq \sum_{j=0}^{k} (\frac{3}{4})^j a_{N_0} < \frac{a_{N_0}}{1-\frac{3}{4}} \Rightarrow \sum_{n=N_0}^{\infty} a_n < \infty.
$$

Therefore, $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

Ratio Test:

Theorem: [Ratio Test]

Given a series $\sum_{n=0}^{\infty} a_n$, assume that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

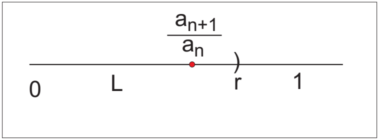
where $L \in \mathbb{R}$ or $L = \infty$.

1. If $0 \le L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.
2. If $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.
3. If $L = 1$, then no conclusion is possible.

Ratio Test:

Proof of 1): Assume that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1.$$



Let $L < r < 1$. Then we can find an N_0 such that if $n \geq N_0$, then

$$0 < \frac{|a_{n+1}|}{|a_n|} < r.$$

We again get that for each $k \in \mathbb{N}$

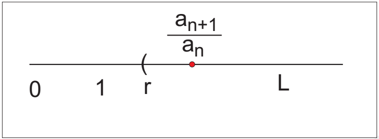
$$|a_{N_0+k}| \leq |a_{N_0}|r^k.$$

Since $0 < r < 1$, the Geometric Series Test shows that $\sum_{k=0}^{\infty} |a_{N_0}|r^k$ converges. The Comparison Test shows that $\sum_{n=N_0}^{\infty} |a_n|$ also converges. Hence, $\sum_{n=1}^{\infty} |a_n|$ converges.

Ratio Test:

Proof of 2): Assume that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1.$$



Let $1 < r < L$. Then we can find an N_0 such that if $n \geq N_0$, then

$$\left|\frac{a_{n+1}}{a_n}\right| > r.$$

If $n \geq N_0$, then

$$|a_{n+1}| \geq r \cdot |a_n| > |a_n|.$$

In fact,

$$|a_{N_0+k}| \geq |a_{N_0}|r^k \to \infty.$$

The Divergence Test shows that $\sum_{n=1}^{\infty} a_n$ diverges.

Ratio Test:

Example : Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. Then

1) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

2) $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Fact: If p and q are polynomials and if

$$a_n = \frac{p(n)}{q(n)},$$

then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

so the Ratio Test fails.

Ratio Test:

Important notes :

1) The Ratio Test only detects convergence if $a_n \to 0$ **very rapidly**.
2) The Ratio Test only detects divergence if $|a_n| \to \infty$.
3) Despite 1) and 2), the Ratio Test is probably the most important test for convergence of series.

Ratio Test:

Question: Does the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converge or diverge?

Solution: Let $a_n = \frac{1}{n!}$. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \\ &= \lim_{n\to\infty} \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{1}{n+1} \\ &= 0. \end{aligned}$$

The Ratio Test shows that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Ratio Test:

Question: Does the series $\sum_{n=0}^{\infty} \frac{1000000^n}{n!}$ converge or diverge?

Note: If we let

$$a_n = \frac{1000000^n}{n!}$$

then

$$a_{10} > 10^{50}$$

However,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1000000^{n+1}}{(n+1)!}}{\frac{1000000^n}{n!}} = \frac{1000000}{n+1}$$

We again have that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1000000}{n+1} = 0$$

so the Ratio Test show that $\sum_{n=0}^{\infty} \frac{1000000^n}{n!}$ converges.

Ratio Test:

Question: For which $x \in \mathbb{R}$ does the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge or diverge?

Solution: Let $a_n = \frac{x^n}{n!}$. Then if $x \neq 0$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} \\
&= \lim_{n\to\infty} \frac{|x|^{n+1} n!}{|x|^n (n+1)!} \\
&= \lim_{n\to\infty} \frac{|x|}{n+1} \\
&= 0.
\end{aligned}$$

The Ratio Test shows that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$.

Important Observation: We have $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ so factorials eventually grow much quicker than exponentials.

Ratio Test:

Question: Does the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converge or diverge?

Solution: Let $a_n = \frac{n^n}{n!}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} \\
&= \lim_{n\to\infty} \frac{(n+1)^{n+1} n!}{n^n (n+1)!} \\
&= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \\
&= \lim_{n\to\infty} (\frac{n+1}{n})^n \\
&= \lim_{n\to\infty} (1+\frac{1}{n})^n \\
&= e > 1.
\end{aligned}$$

The Ratio Test shows that $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

Question: Does the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge or diverge