Positive Series

Created by

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Positive Series

Definition: [Positive Series]

A series $\sum_{n=1}^{\infty} a_n$ is called *positive* if $a_n \geq 0$ for all $n \in \mathbb{N}$.

Note: If $\sum_{n=1}^{\infty} a_n$ is a positive series, then

$$\begin{aligned}
S_{k+1} - S_k &= \sum_{n=1}^{k+1} a_n - \sum_{n=1}^{k} a_n \\
&= (a_1 + a_2 + \cdots + a_k + a_{k+1}) - (a_1 + a_2 + \cdots + a_k) \\
&= (\cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_k} + a_{k+1}) - (\cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_k}) \\
&= a_{k+1} \\
&\geq 0
\end{aligned}$$

Hence $\{S_k\}$ is a nondecreasing sequence.

Positive Series

Key Observation:

If $\sum_{n=1}^{\infty} a_n$ is a positive series with partial sums $S_k = \sum_{n=1}^{k} a_n$, then

$$\text{MCT} \Rightarrow \begin{cases} \{S_k\} \text{ converges} & \text{if } \{S_k\} \text{ is bounded,} \\ \{S_k\} \text{ diverges to } \infty & \text{otherwise.} \end{cases}$$

$$\text{MCT} \Rightarrow \begin{cases} \sum_{n=1}^{\infty} a_n \text{ converges} & \text{if } \{S_k\} \text{ is bounded,} \\ \sum_{n=1}^{\infty} a_n \text{ diverges to } \infty & \text{otherwise.} \end{cases}$$

Example

Example: Show that $\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ converges.

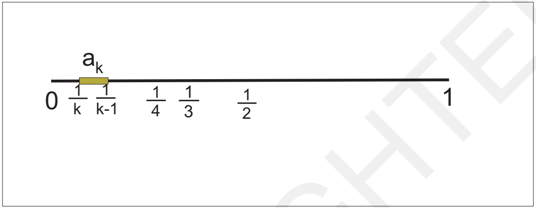
Note that

$$a_n = \frac{1}{n^2 - n} = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

so

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

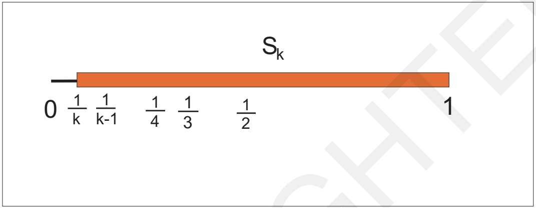
Example



Geometric Solution:

$$
\begin{aligned}
a_2 &= \frac{1}{1} - \frac{1}{2} \rightarrow \text{distance from } 1 \text{ to } \frac{1}{2}. \\
a_3 &= \frac{1}{2} - \frac{1}{3} \rightarrow \text{distance from } \frac{1}{2} \text{ to } \frac{1}{3}. \\
a_4 &= \frac{1}{3} - \frac{1}{4} \rightarrow \text{distance from } \frac{1}{3} \text{ to } \frac{1}{4}. \\
&\vdots \\
a_k &= \frac{1}{k-1} - \frac{1}{k} \rightarrow \text{distance from } \frac{1}{k-1} \text{ to } \frac{1}{k}.
\end{aligned}
$$

Example



$$
\begin{aligned}
S_2 &= 1 - \frac{1}{2} \\
S_3 &= S_2 + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3} \\
S_4 &= S_3 + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4} \\
&\vdots \\
S_k &= S_{k-1} + \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 - \frac{1}{k}
\end{aligned}
$$

Example

Algebraic Solution:

$$
\begin{aligned}
S_k &= (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) \\
&\quad \cdots + (\frac{1}{k-2}-\frac{1}{k-1}) + (\frac{1}{k-1}-\frac{1}{k}) \\
&= (1-\cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}) + (\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}) \\
&\quad + \cdots + (\cancel{\frac{1}{k-2}}-\cancel{\frac{1}{k-1}}) + (\cancel{\frac{1}{k-1}}-\frac{1}{k}) \\
&= 1-\frac{1}{k}
\end{aligned}
$$

Example

Conclusion: The series

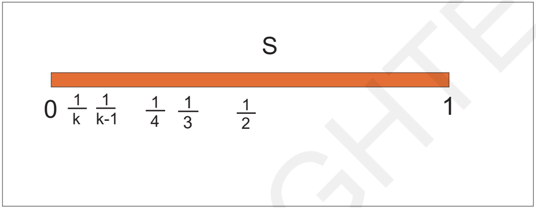
$$\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$$

is positive and

$$S_k = 1 - \frac{1}{k} < 1$$

for each $k \geq 2$, hence the series converges.

Example



We have

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{1}{n^2-n} &= \lim_{k\to\infty}\sum_{n=2}^{k}\frac{1}{n^2-n}\\ &= \lim_{k\to\infty} 1-\frac{1}{k}\\ &= 1\end{aligned}$$

Example

Problem: Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

Key Observation: If $n \geq 2$, then

$$\frac{1}{n^2} < \frac{1}{n^2 - n}.$$

Let $T_k = \sum_{n=1}^{k} \frac{1}{n^2}$ and $S_k = \sum_{n=2}^{k} \frac{1}{n^2-n}$. If $k \geq 2$,

$$\begin{aligned} T_k &= \sum_{n=1}^{k} \frac{1}{n^2} \\ &= 1 + \sum_{n=2}^{k} \frac{1}{n^2} \\ &\leq 1 + \sum_{n=2}^{k} \frac{1}{n^2 - n} \\ &= 1 + S_k \\ &\leq 1 + 1 = 2. \end{aligned}$$

$\Rightarrow$ T_k is bounded above by 2.
$\Rightarrow$ $\{T_k\}$ converges.
$\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \cong 1.64493$$