The Limit Comparison Test

Created by

Barbara Forrest and Brian Forrest

Failure of the Comparison Test

Question : Does the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

converge?

We know that

$$0 < \sin\left(\frac{1}{n}\right) \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$ but $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges so the Comparison Test fails.

Since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, the Fundamental Trig Limits shows that

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

so for large n we have

$$\sin\left(\frac{1}{n}\right) \cong \frac{1}{n}$$

Does this mean that

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

also diverges?

The Limit Comparison Test

Theorem: [The Limit Comparison Test for Series]

Assume that $a_n > 0$ and $b_n > 0$ for each $n \in \mathbb{N}$. Assume also that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where either $L \in \mathbb{R}$ or $L = \infty$.

1) If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

2) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   Equivalently, if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

3) If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

   Equivalently, if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

The Limit Comparison Test

Proof of the Limit Comparison Test: First we assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

1) If $0 < L < \infty$, the interval $(\frac{L}{2}, 2L)$ is an open interval containing L. It follows that we can find a cutoff $N \in \mathbb{N}$ so that if $n \geq N$, then

$$\frac{L}{2} < \frac{a_n}{b_n} < 2L$$

or equivalently that

$$\frac{L}{2} \cdot b_n < a_n < 2Lb_n$$

If $\sum_{n=1}^{\infty} a_n$ converges, then the Comparison Test shows that

$$\sum_{n=1}^{\infty} \frac{L}{2} \cdot b_n$$

converges and hence so does

$$\sum_{n=1}^{\infty} b_n$$

The Limit Comparison Test

Proof of the Limit Comparison Test (continued):

1) Since

$$\frac{L}{2} \cdot b_n < a_n < 2Lb_n,$$

if $\sum_{n=1}^{\infty} b_n$ converges, then so does

$$\sum_{n=1}^{\infty} 2L \cdot b_n$$

By the Comparison Test

$$\sum_{n=1}^{\infty} a_n$$

also converges.

The Limit Comparison Test

Proof of the Limit Comparison Test (continued):

2) If $L = 0$, then we can find a cut off $N \in \mathbb{N}$ so that if $n \geq N$, then

$$0 < \frac{a_n}{b_n} < 1$$

or equivalently that

$$0 < a_n < b_n$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well by the Comparison Test.

Equivalently, if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

The Limit Comparison Test

Proof of the Limit Comparison Test (continued):

3) If $L = \infty$, then we can find a cut off $N \in \mathbb{N}$ so that if $n \geq N$, then

$$\frac{a_n}{b_n} > 1$$

or equivalently that

$$b_n < a_n$$

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges as well by the Comparison Test.

Equivalently, if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

The Limit Comparison Test

Summary:

1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$, then for large n we have

$$\frac{a_n}{b_n} \cong L$$

or

$$a_n \cong L b_n.$$

When $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$, we say that a_n *and* b_n *have the same order of magnitude*. We write

$$a_n \approx b_n$$

The Limit Comparison Test says that **two positive series with terms of the same order of magnitude will have the same convergence properties.**

The Limit Comparison Test

Summary:

2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then b_n must eventually be much larger than a_n.

   In this case, we write

$$a_n \ll b_n$$

   and we say that *the order of magnitude of* a_n *is smaller than the order of magnitude of* b_n.

   If the smaller series $\sum_{n=1}^{\infty} a_n$ diverges to ∞, it would make sense that $\sum_{n=1}^{\infty} b_n$ also diverges to ∞.

3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then a_n must eventually be much larger than b_n.

   That is $b_n \ll a_n$. This time, if the larger series $\sum_{n=1}^{\infty} a_n$ converges, it would make sense that $\sum_{n=1}^{\infty} b_n$ would converge as well.

Example: The series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges.

Since

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$$

and since

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges the Limit Comparison Test shows that

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

also diverges.

Example: The series $\sum_{n=1}^{\infty} \frac{2n}{n^3-n+1}$ converges.

Let $a_n = \frac{2n}{n^3-n+1}$ and $b_n = \frac{1}{n^2}$. Then

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{\frac{2n}{n^3-n+1}}{\frac{1}{n^2}} \\ &= \frac{2n^3}{n^3-n+1} \\ &= \frac{n^3}{n^3}\left(\frac{2}{1-\frac{1}{n^2}+\frac{1}{n^3}}\right) \\ &= \frac{2}{1-\frac{1}{n^2}+\frac{1}{n^3}} \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2}{1-\frac{1}{n^2}+\frac{1}{n^3}} = \frac{2}{1} = 2.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the Limit Comparison Test shows that $\sum_{n=1}^{\infty} \frac{2n}{n^3-n+1}$ converges.