Introduction to Series

Created by

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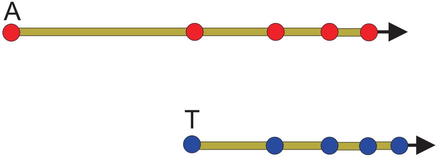
A Problem to Consider

Problem:

Can infinitely many tasks be performed in a finite amount of time?

The Paradox of Achilles

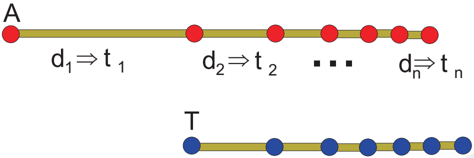


Paradox of Achilles

- Achilles races a tortoise who is given a head start.
- Achilles **reaches the point where the tortoise began**, but the tortoise has moved ahead to a new point.
- Achilles **reaches the new point**. Again, the tortoise has moved ahead.

- Achilles reaches the next point, and again the tortoise has moved ahead.
- And so on . . ..
- **Conclusion:** Achilles can never catch the tortoise!!!

Resolving The Paradox of Achilles



Resolving the Paradox

We call each time Achilles moves to where the tortoise was a *stage*.

- d_1 = distance Achilles traveled in stage 1 $\Rightarrow$ t_1 = time to complete stage 1
- d_2 = distance Achilles traveled in stage 2 $\Rightarrow$ t_2 = time to complete stage 2
- d_n = distance Achilles traveled in stage n $\Rightarrow$ t_n = time to complete stage n

$$\begin{aligned}\text{Time to catch the Tortoise} &= t_1 + t_2 + \cdots + t_n + \cdots \\ &= \infty\ ?\end{aligned}$$

Introduction to Series

Problem:

Can we add infinitely many numbers at the same time?

More precisely, given a sequence $\{a_n\}$, we can form the *formal sum*

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

which is called a *series*?

Question:

What does this formal sum represent? Does it have a value?

Introduction to Series

Example: What is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots ?$$

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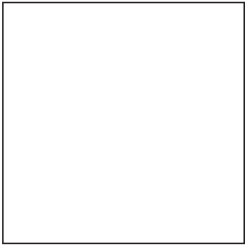
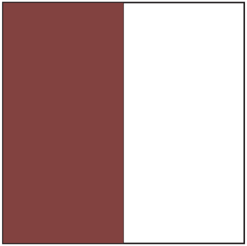


Geometric Interpretation

Introduction to Series

Geometric Interpretation



$$\frac{1}{2} +$$

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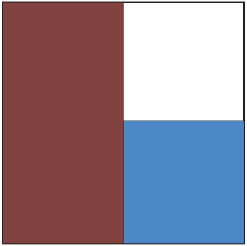

$$\frac{1}{2} + \frac{1}{4} +$$

Geometric Interpretation

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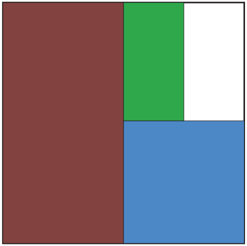

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$$

Geometric Interpretation

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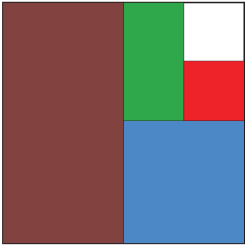

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$$

Geometric Interpretation

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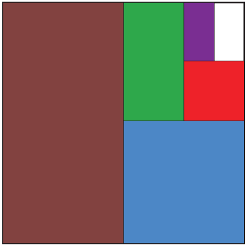
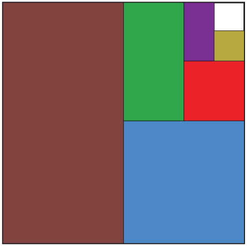


$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+$$

Geometric Interpretation

Introduction to Series



Geometric Interpretation

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+$$

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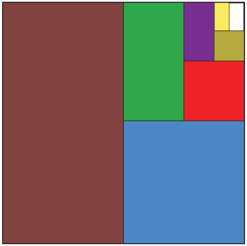

Geometric Interpretation

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} +$$

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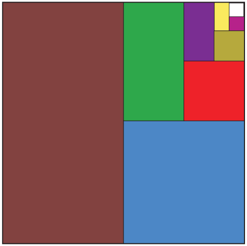

Geometric Interpretation

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+$$

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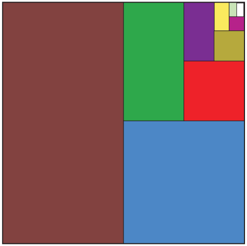

Geometric Interpretation

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+$$

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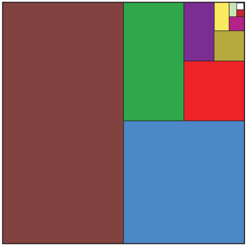

Geometric Interpretation

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}+$$

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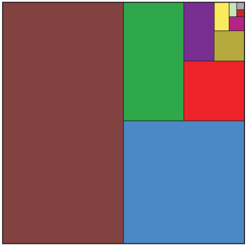

Geometric Interpretation

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}+\cdots$$

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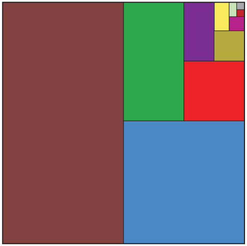

Geometric Interpretation

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \cdots$$

$$= 1?$$

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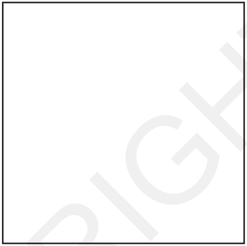


Introduction to Series

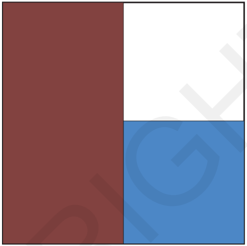
Sum of Areas

$\frac{1}{2}$

Total Area Covered

$1 - \frac{1}{2}$

Introduction to Series



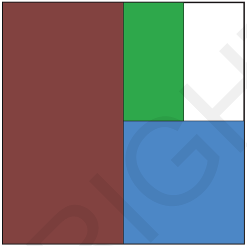
Sum of Areas

$\frac{1}{2} + \frac{1}{4}$

Total Area Covered

$1 - \frac{1}{4}$

Introduction to Series



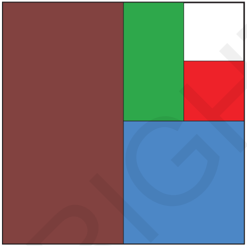
Sum of Areas

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

Total Area Covered

$1 - \frac{1}{8}$

Introduction to Series



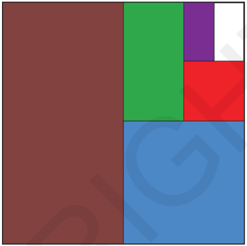
Sum of Areas

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

Total Area Covered

$1 - \frac{1}{16}$

Introduction to Series



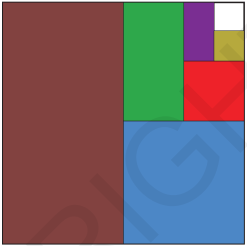
Sum of Areas

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$$

Total Area Covered

$$1 - \frac{1}{32}$$

Introduction to Series



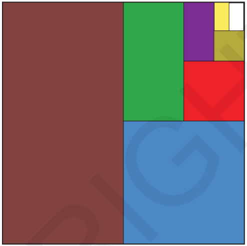
Sum of Areas

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$$

Total Area Covered

$$1 - \frac{1}{64}$$

Introduction to Series



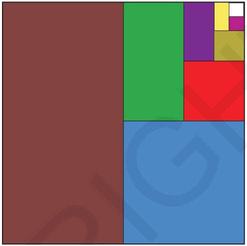
Sum of Areas

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$$

Total Area Covered

$$1 - \frac{1}{128}$$

Introduction to Series



Sum of Areas

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}$$

Total Area Covered

$$1 - \frac{1}{256}$$

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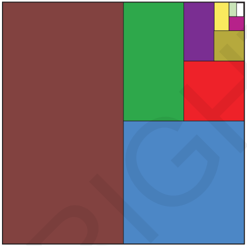


Sum of Areas

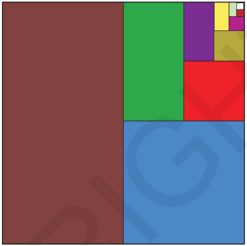
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$$

$$+\frac{1}{256} + \frac{1}{512}$$

Total Area Covered

$$1 - \frac{1}{512}$$

Introduction to Series



Sum of Areas

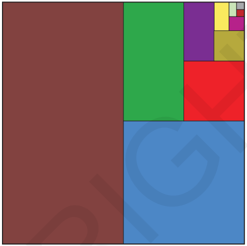
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$$

$$+ \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$$

Total Area Covered

$$1 - \frac{1}{1024}$$

Introduction to Series



Sum of Areas

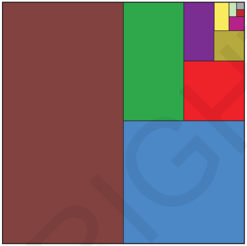
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}$$

$$+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}+\cdots+\frac{1}{2^k}$$

Total Area Covered

$$1-\frac{1}{2^k}$$

Introduction to Series



Note:

$$\lim_{k\to\infty} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{2^n}$$

$$= \lim_{k\to\infty} 1 - \frac{1}{2^k}$$

$$= 1$$

Series

Definition: [Series]

Given a sequence $\{a_n\}$, the *formal* sum

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

is called a *series*. (The series is called *formal* because we have not yet given it a meaning numerically.)

The a_n's are called the *terms* of the series. For each term a_n, the number n is called the *index* of the term.

We denote the series by

$$\sum_{n=1}^{\infty} a_n.$$

Convergent/Divergent Series

Definition: [Convergent Series]

Given a sequence $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$, we define the *$k$th partial sum S_k* of the series $\sum_{n=1}^{\infty} a_n$ by

$$S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ *converges* if the sequence of partial sums $\{S_k\}$ converges. In this case, we write

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k$$

Otherwise, we say that the series *diverges* and the sum has no defined value.

Example

Example:

Suppose $a_n = \frac{1}{2^n}$. We know that

$$S_k = \sum_{n=1}^{k} \frac{1}{2^n} = 1 - \frac{1}{2^k} \to 1.$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges with

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Why Use Limits?

Question: Why use limits?

Suppose $a_n = (-1)^{n+1}$. Then the formal sum looks like

$$a_1 + a_2 + a_3 + \cdots = 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$$

We could parenthesize the formal sum as follows:

$$[1 + (-1)] + [1 + (-1)] + [1 + (-1)] + \cdots = 0 + 0 + 0 + \cdots = 0.$$

Alternatively, we could parenthesize the formal sum as:

$$1 + [(-1) + 1] + [(-1) + 1] + [(-1) + 1] + \cdots = 1 + 0 + 0 + 0 + \cdots = 1.$$

Our result is ambiguous; the "sum" changes if we change the way we parenthesize the terms!

Why Use Limits?

Observe:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 - 1 = 0 \\ S_3 &= 1 - 1 + 1 = 1 \\ S_4 &= 1 - 1 + 1 - 1 = 0 \end{aligned}$$

We get

$$S_k = 1 + (-1) + 1 + \cdots + (-1)^{k+1} = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Thus, $\{S_k\} = \{1, 0, 1, 0, 1, 0, \cdots\}$ **diverges**.

Example

Example: Determine if the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

converges or diverges.

Solution: Observe that

$$a_n = \frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

so the series becomes

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

We have

$$\begin{aligned} S_k &= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{k} - \frac{1}{k+1}) \\ &= 1 - (\frac{1}{2} - \frac{1}{2}) - (\frac{1}{3} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{4}) - \cdots - (\frac{1}{k} - \frac{1}{k}) - \frac{1}{k+1} \\ &= 1 - 0 - 0 - 0 - \cdots - 0 - \frac{1}{k+1} = 1 - \frac{1}{k+1} \to 1 \end{aligned}$$