

The Integral Test Part 2: p-Series Test

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Theorem: [The Integral Test]

Assume that

- f is continuous on $[1, \infty)$,
- $f(x) > 0$ on $[1, \infty)$,
- f is decreasing on $[1, \infty)$, and
- $a_k = f(k)$.

Then

- 1) If $S_n = \sum_{k=1}^n a_k$, then for all $n \in \mathbb{N}$,

$$\int_1^{n+1} f(x) dx \leq S_n \leq a_1 + \int_1^n f(x) dx.$$

- 2) $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_1^{\infty} f(x) dx$ converges.

- 3) If $\sum_{k=1}^{\infty} a_k$ converges, with $S = \sum_{k=1}^{\infty} a_k$, then

$$\int_1^{\infty} f(x) dx \leq \sum_{k=1}^{\infty} a_k \leq a_1 + \int_1^{\infty} f(x) dx$$

and

$$\int_{n+1}^{\infty} f(x) dx \leq S - S_n \leq \int_n^{\infty} f(x) dx.$$

p-Series Test

Key Observation : The function $f(x) = \frac{1}{x^p}$ satisfies the conditions of the Integral Test for $p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p} dx$$

converges if and only if $p > 1$.

Theorem: [p-Series Test]

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Example

Example: Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

converges.

Observation: Let $f(x) = \frac{1}{x \ln(x)}$ and let $a_n = f(n) = \frac{1}{n \ln(n)}$. Then f is continuous, positive, and decreasing on $[2, \infty)$.

Hence

$$\sum_{n=2}^{\infty} a_n$$

converges if and only if

$$\int_2^{\infty} f(x) dx$$

converges.

Example

Example (continued): We compute

$$\int_2^{\infty} \frac{1}{x \ln(x)} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x \ln(x)} dx.$$

Make a change of variable $u = \ln(x)$, and so $du = \frac{1}{x} dx$ and

$$\begin{aligned} \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x \ln(x)} dx &= \lim_{b \rightarrow \infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u} du \\ &= \lim_{b \rightarrow \infty} \ln(u) \Big|_{\ln(2)}^{\ln(b)} \\ &= \lim_{b \rightarrow \infty} \ln(\ln(b)) - \ln(\ln(2)) \end{aligned}$$

which diverges to ∞ . Therefore

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

diverges by the Integral Test.

Example

Example:

Show that $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges.

Observation: Since $f(x) = \frac{1}{x(\ln(x))^2}$ is continuous and decreasing with $f(x) > 0$ on $[2, \infty)$, the Integral Test can be used to conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

converges if and only if

$$\int_2^{\infty} \frac{1}{x(\ln(x))^2} dx$$

converges.

Example

Example (continued): To evaluate $\int_2^b \frac{1}{x(\ln(x))^2} dx$, use the substitution $u = \ln(x)$, $du = \frac{dx}{x}$ to get

$$\begin{aligned}\int_2^\infty \frac{1}{x(\ln(x))^2} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln(x))^2} dx \\ &= \lim_{b \rightarrow \infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u^2} du \\ &= \lim_{b \rightarrow \infty} \left. -\frac{1}{u} \right|_{\ln(2)}^{\ln(b)} \\ &= \lim_{b \rightarrow \infty} \frac{1}{\ln(2)} - \frac{1}{\ln(b)} \\ &= \frac{1}{\ln(2)}\end{aligned}$$

Since $\int_2^\infty \frac{1}{x(\ln(x))^2} dx$ converges, so does $\sum_{n=2}^\infty \frac{1}{n(\ln(n))^2}$.