The Integral Test Part 1: Introduction

Created by

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p-Series Revisited

Recall: We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges while $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

We also know that if $1 < p < 2$, then

$$\frac{1}{n^2} \leq \frac{1}{n^p} \leq \frac{1}{n}$$

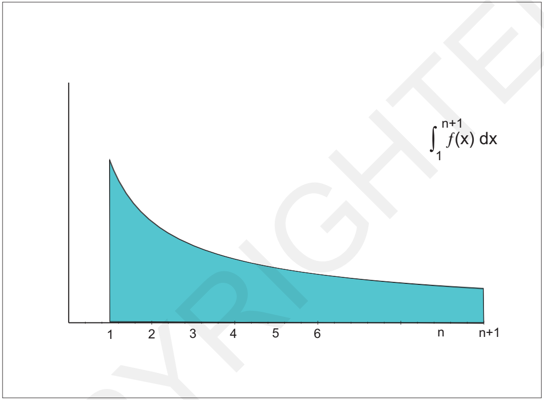
so the Comparison Test (and Limit Comparison Test) fail to determine if

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges.

Key Idea: There is a close relationship between series and improper integrals.

Integral Test



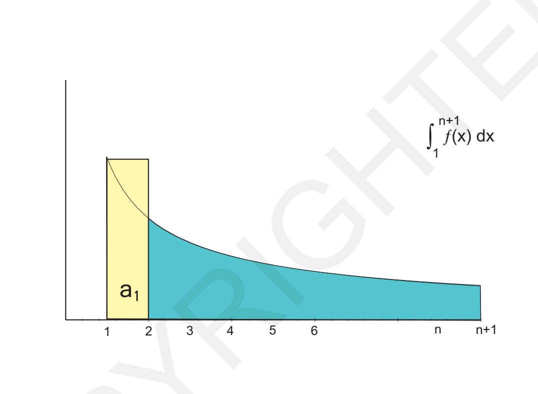
Assume that f is continuous, decreasing and $f(x) > 0$ on $[1, \infty)$. For each $k \in \mathbb{N}$, let

$$a_k = f(k) \quad \textbf{and} \quad S_n = \sum_{k=1}^{n} a_k$$

We claim

$$\int_1^{n+1} f(x)\,dx \le S_n \le a_1 + \int_1^{n} f(x)\,dx.$$

Integral Test



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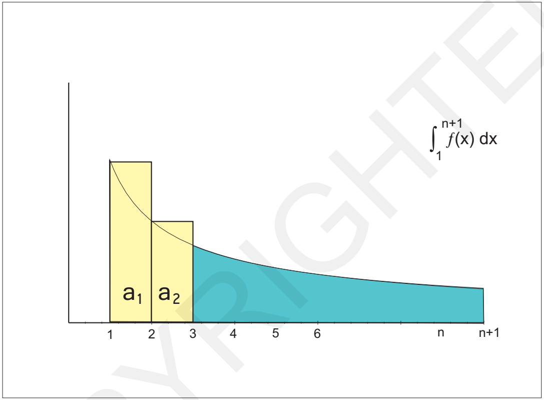

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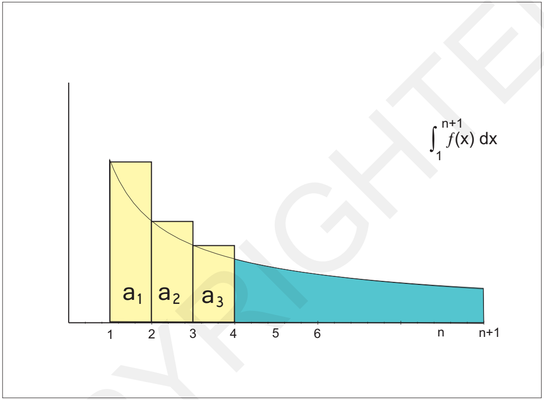

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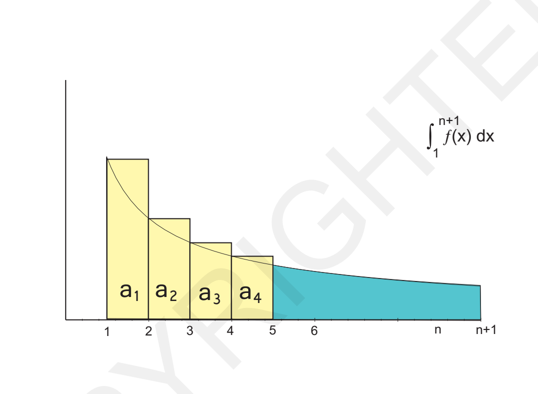

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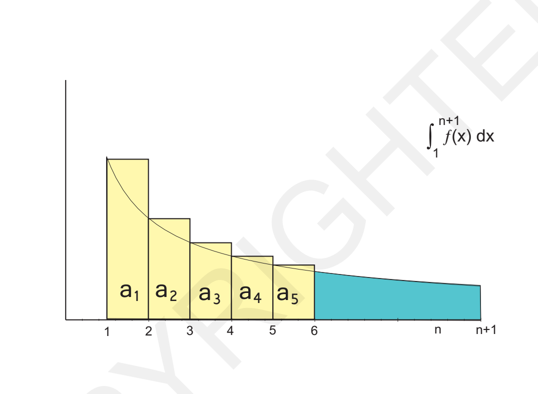

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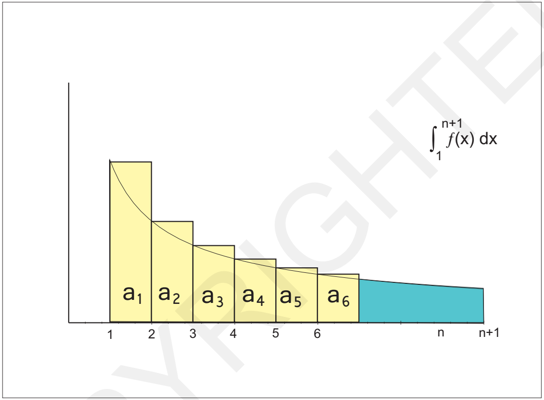

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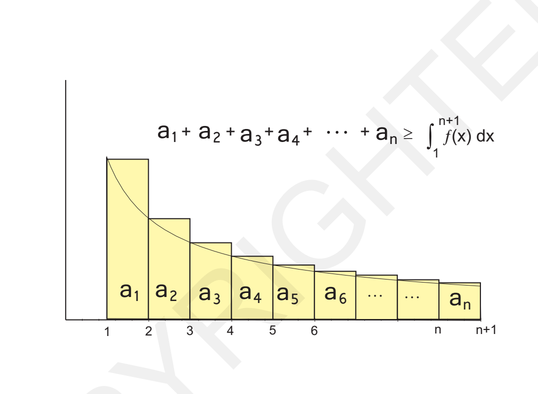

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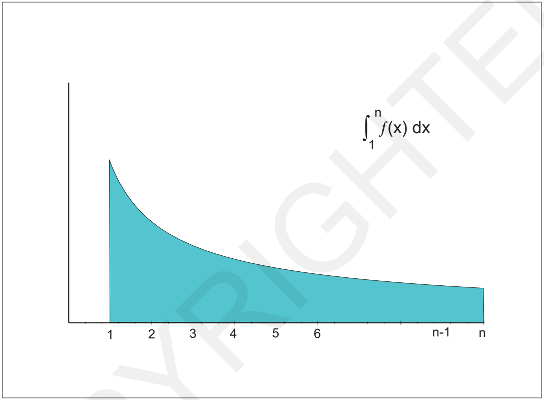

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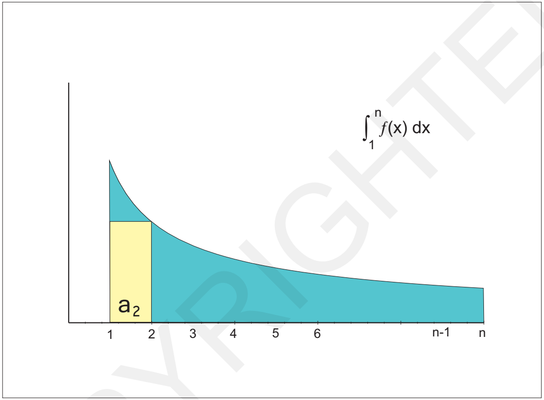

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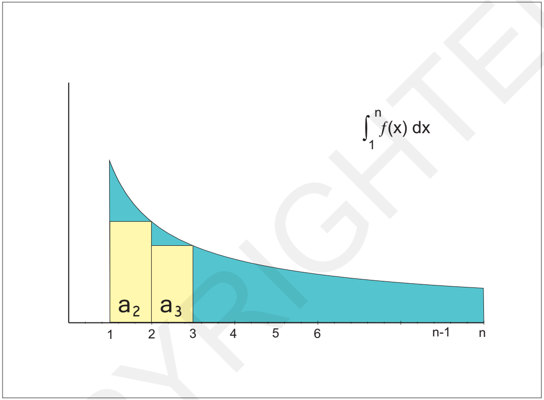

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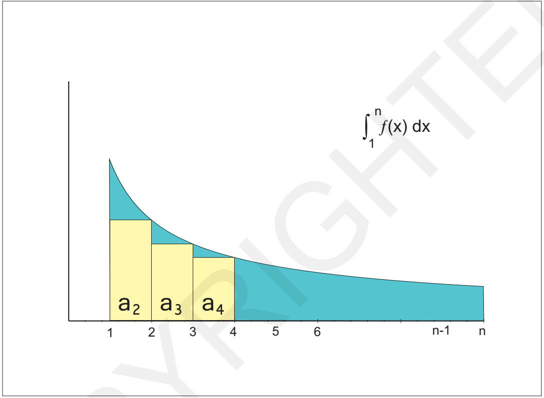

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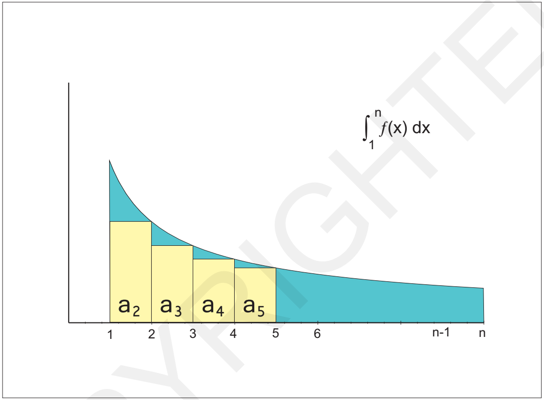

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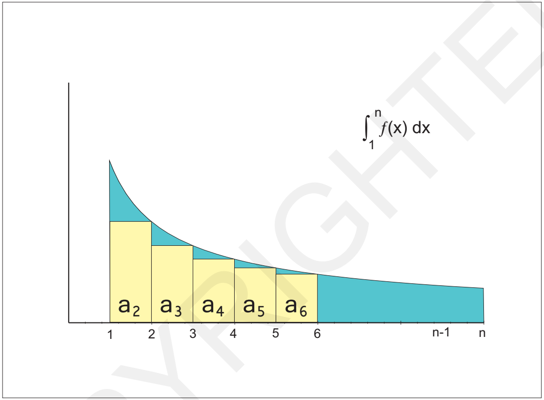

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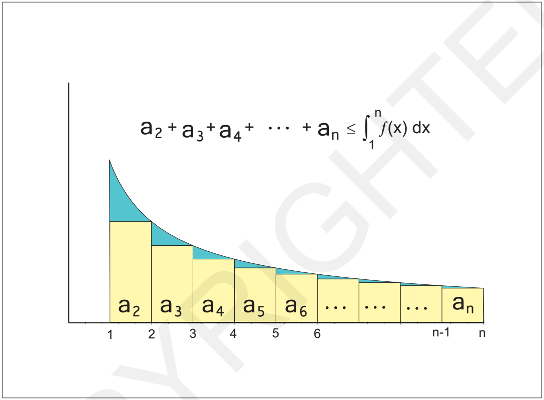

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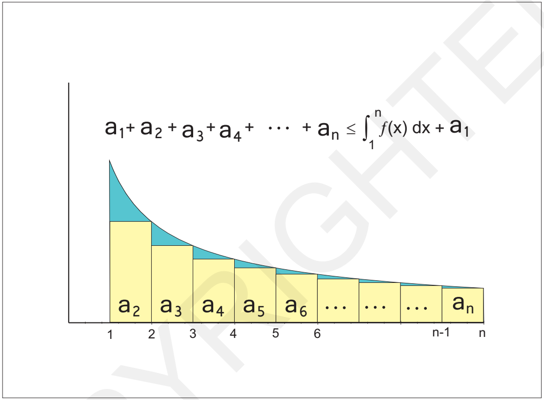

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Integral Test

Theorem: [The Integral Test]

Assume that

f is continuous on $[1, \infty)$,
$f(x) > 0$ on $[1, \infty)$,
f is decreasing on $[1, \infty)$, and
$a_k = f(k)$.

Then

1) If $S_n = \sum_{k=1}^{n} a_k$, then for all $n \in \mathbb{N}$,

$$\int_1^{n+1} f(x)\,dx \le S_n \le a_1 + \int_1^n f(x)\,dx.$$

2) $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

3) If $\sum_{k=1}^{\infty} a_k$ converges, with $S = \sum_{k=1}^{\infty} a_k$, then

$$\int_1^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} a_k \le a_1 + \int_1^{\infty} f(x)\,dx$$

and

$$\int_{n+1}^{\infty} f(x)\,dx \le S - S_n \le \int_n^{\infty} f(x)\,dx.$$

Integral Test

Note:

1) We have already established that

$$\int_1^{n+1} f(x)\,dx \le S_n \le a_1 + \int_1^n f(x)\,dx.$$

2) If $\sum_{k=1}^{\infty} a_k$ converges, then

$$\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{\infty} a_k$$

for all $n \in \mathbb{N}$, so

$$\int_1^{\infty} f(x)\,dx$$

converges by the Monotone Convergence Theorem for Functions.

If $\int_1^{\infty} f(x)\,dx$ converges, then

$$\sum_{k=1}^{n} a_k \le a_1 + \int_1^{\infty} f(x)\,dx$$

for all $n \in \mathbb{N}$, so

$$\sum_{k=1}^{\infty} a_k$$

converges by the Monotone Convergence Theorem.

Integral Test

Note:

3) If $\sum_{k=1}^{\infty} a_k$ converges, with $S = \sum_{k=1}^{\infty} a_k$, then

$$\lim_{n\to\infty} \int_1^{n+1} f(x)\,dx \le \lim_{n\to\infty} \sum_{k=1}^{n} a_k \le a_1 + \lim_{n\to\infty} \int_1^{n} f(x)\,dx$$

Hence

$$\int_1^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} a_k \le a_1 + \int_1^{\infty} f(x)\,dx.$$

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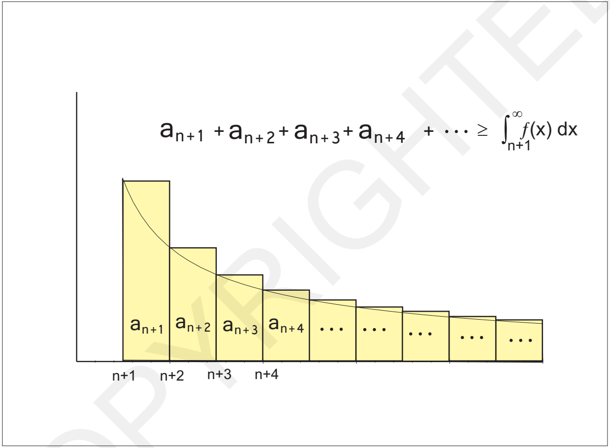

Finally

$$\int_{n+1}^{\infty} f(x)\,dx \leq S - S_n \leq \int_{n}^{\infty} f(x)\,dx.$$

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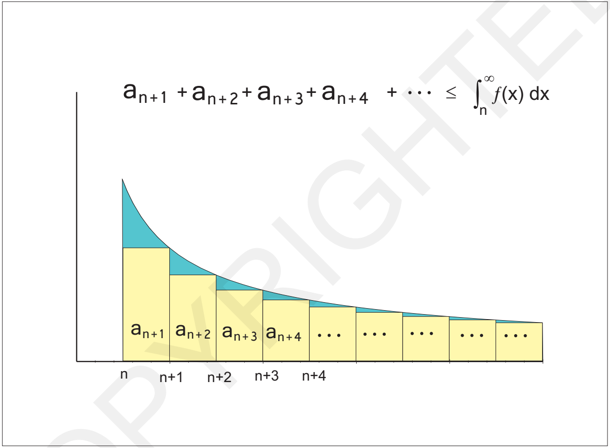

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p-Series Test

Key Observation : The function $f(x) = \frac{1}{x^p}$ satisfies the conditions of the Integral Test for $p > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges if and only if $p > 1$.

Theorem: [p-Series Test]

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Example

Example: Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$$

converges.

Observation: Let $f(x) = \frac{1}{x\ln(x)}$ and let $a_n = f(n) = \frac{1}{n\ln(n)}$. Then f is continuous, positive, and decreasing on $[2, \infty)$.

Hence

$$\sum_{n=2}^{\infty} a_n$$

converges if and only if

$$\int_2^{\infty} f(x)\,dx$$

converges.

Example

Example (continued): We compute

$$\int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln(x)}\,dx.$$

Make a change of variable $u = \ln(x)$, and so $du = \frac{1}{x}\,dx$ and

$$\begin{aligned}\lim_{b\to\infty}\int_2^b \frac{1}{x\ln(x)}\,dx &= \lim_{b\to\infty}\int_{\ln(2)}^{\ln(b)} \frac{1}{u}\,du \\ &= \lim_{b\to\infty} \ln(u)\Big|_{\ln(2)}^{\ln(b)} \\ &= \lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(2))\end{aligned}$$

which diverges to ∞. Therefore

$$\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$$

diverges by the Integral Test.

Example

Example:

Show that $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges.

Observation: Since $f(x) = \frac{1}{x(\ln(x))^2}$ is continuous and decreasing with $f(x) > 0$ on $[2, \infty)$, the Integral Test can be used to conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

converges if and only if

$$\int_2^{\infty} \frac{1}{x(\ln(x))^2}\, dx$$

converges.

Example

Example (continued): To evaluate $\int_2^b \frac{1}{x(\ln(x))^2}\,dx$, use the substitution $u = \ln(x)$, $du = \frac{dx}{x}$ to get

$$\begin{aligned}
\int_2^\infty \frac{1}{x(\ln(x))^2}\,dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln(x))^2}\,dx \\
&= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u^2}\,du \\
&= \lim_{b\to\infty} -\frac{1}{u}\Big|_{\ln(2)}^{\ln(b)} \\
&= \lim_{b\to\infty} \frac{1}{\ln(2)} - \frac{1}{\ln(b)}. \\
&= \frac{1}{\ln(2)}
\end{aligned}$$

Since $\int_2^\infty \frac{1}{x(\ln(x))^2}\,dx$ converges, so does $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$.