The Comparison Test for Series

Created by

Barbara Forrest and Brian Forrest

Example

Problem: Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

Key Observation: We saw that

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 1$$

and that if $n \geq 2$, then

$$\frac{1}{n^2} < \frac{1}{n^2 - n}.$$

If

$$T_k = \sum_{n=1}^{k} \frac{1}{n^2} \text{ and } S_k = \sum_{n=2}^{k} \frac{1}{n^2 - n}$$

then for $k \geq 2$,

$$T_k = 1 + \sum_{n=2}^{k} \frac{1}{n^2} \leq 1 + \sum_{n=2}^{k} \frac{1}{n^2 - n} < 2$$

Hence $\{T_k\}$ is bounded above by 2 and $\{T_k\}$ converges, as does

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Comparison Test

Question: If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, must $\sum_{n=1}^{\infty} a_n$ converge?

Theorem: [The Comparison Test for Series]

Assume that $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$.

1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Comparison Test

Proof of the Comparison Test:

Assume that $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$. Let

$$T_k = \sum_{n=1}^{k} a_n \quad \text{and} \quad S_k = \sum_{n=1}^{k} b_n$$

1) Assume that $\sum_{n=1}^{\infty} b_n$ converges with $\sum_{n=1}^{\infty} b_n = S$. Then

$$T_k = \sum_{n=1}^{k} a_n \leq S_k = \sum_{n=1}^{k} b_n \leq S$$

for each $k \in \mathbb{N}$. It follows that $\{T_k\}$ is bounded and that

$$\sum_{n=1}^{\infty} a_n$$

also converges.

Comparison Test

Proof of the Comparison Test:

Assume that $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$. Let

$$T_k = \sum_{n=1}^{k} a_n \text{ and } S_k = \sum_{n=1}^{k} b_n$$

2) Assume that $\sum_{n=1}^{\infty} a_n$ diverges. Let $M > 0$. Then we can find an $N \in \mathbb{N}$ so that $M \leq T_N$. But if $k \geq N$,

$$M \leq T_N \leq S_N \leq S_k$$

and hence

$$\sum_{n=1}^{\infty} b_n$$

also diverges.

Comparison Test

Three Important Observations:

1) If $\sum_{n=1}^{\infty} a_n$ converges, then we cannot say anything about $\sum_{n=1}^{\infty} b_n$.
2) If $\sum_{n=1}^{\infty} b_n$ diverges, then we cannot say anything about $\sum_{n=1}^{\infty} a_n$.
3) Since the first few terms do not affect whether or not a series diverges, for the Comparison Test to hold, we really only need that

$$0 \leq a_n \leq b_n$$

for each $n \geq K$, where $K \in \mathbb{N}$. That is, the conditions of the theorem need only be satisfied by the elements of the tails of the two sequences.

Examples

Example: We have seen that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and that

$$0 < \frac{1}{n^3} \le \frac{1}{n^2}$$

for all $n \in \mathbb{N}$.
If $a_n = \frac{1}{n^3}$ and $b_n = \frac{1}{n^2}$, then the Comparison Test shows that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

also converges.
In fact, if $p \ge 2$, then

$$0 < \frac{1}{n^p} \le \frac{1}{n^2}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges.

Examples

Note: We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and that

$$0 < \frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

for all $n \in \mathbb{N}$.
If $a_n = \frac{1}{n}$ and $b_n = \frac{1}{\sqrt{n}}$, then the Comparison Test shows that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

also diverges.
In fact, if $p \leq 1$, then

$$0 < \frac{1}{n} \leq \frac{1}{n^p}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

diverges.

Examples

Question: We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$ and it converges if $p \geq 2$. What happens if

$$1 < p < 2$$

Note: If $1 < p < 2$, then

$$\frac{1}{n^2} \leq \frac{1}{n^p} \leq \frac{1}{n}$$

so the Comparison Test fails!

Examples

Example: Show that the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$$

converges.

We know that $0 < \sin(x) < x$ for all $x \in (0, 1)$ so

$$0 < \sin\left(\frac{1}{n^2}\right) < \frac{1}{n^2}$$

for all $n \in \mathbb{N}$.

If $a_n = \sin\left(\frac{1}{n^2}\right)$ and $b_n = \frac{1}{n^2}$, then the Comparison Test shows that the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$$

also converges.

Examples

Question: Does the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

converge?

We know that

$$0 < \sin(\frac{1}{n}) < \frac{1}{n}$$

for all $n \in \mathbb{N}$ but $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges so the Comparison Test fails.

Since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, the Fundamental Trig Limits shows that

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

so for large n we have

$$\sin\left(\frac{1}{n}\right) \cong \frac{1}{n}$$

Does this mean that

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

also diverges?