Absolute vs Conditional Convergence

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Important Observation:

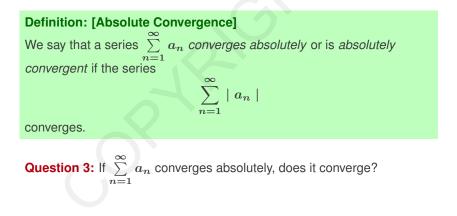
1) The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, while the Alternating Series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges even though the terms have the same order of magnitude,

$$\mid \frac{1}{n} \mid = \frac{1}{n} = \mid (-1)^{n-1} \frac{1}{n} \mid$$

2) On the other hand, both $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ converge because $\frac{1}{n^2}$ is *small enough*!

Question 1: Is there a way to detect if a series will converge because its terms are *small enough*?

Question 2: Aside from the AST all of our tests apply to positive series. Is there any additional method for determining if an arbitrary series $\sum_{n=1}^{\infty} a_n$ converges?



Theorem: [Absolute Convergence Theorem]

Assume that the series $\sum\limits_{n=1}^{\infty}a_n$ converges absolutely. Then $\sum\limits_{n=1}^{\infty}a_n$ converges.

Proof: Assume that

$$\sum_{n=1}^{\infty} \mid a_n \mid$$

converges. Then so does $\sum\limits_{n=1}^{\infty} 2 \mid a_n \mid$. Let

$$b_n = a_n + \mid a_n \mid \ \Rightarrow 0 \leq b_n \leq 2 \mid a_n \mid .$$

By the Comparison Test $\sum_{n=1}^{\infty} b_n$ converges. Since $a_n = b_n - |a_n|$, it follows that $\sum_{n=1}^{\infty} a_n$ also converges and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

Example 1: Let $\{a_n\} = \{1, \frac{1}{2^2}, \frac{-1}{3^2}, \frac{1}{4^2}, \frac{1}{5^2}, \frac{-1}{6^2}, \cdots\}$. Then

$$\mid a_n \mid = rac{1}{n^2}$$

so $\sum\limits_{n=1}^{\infty} \mid a_n \mid$ converges and the series $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.

Example:

Example 2: Let
$$\{b_n\} = \{(-1)^{n+1} \sin(\frac{1}{\sqrt{n}})\}$$
.
Since

1)
$$\sin(\frac{1}{\sqrt{n}}) > 0$$
 for all $n \in \mathbb{N}$,
2) $\sin(\frac{1}{\sqrt{n+1}}) < \sin(\frac{1}{\sqrt{n}})$ for all $n \in \mathbb{N}$,

3)
$$\lim_{n \to \infty} \sin(\frac{1}{\sqrt{n}}) = 0$$
,

the series $\sum_{n=1}^{\infty} (-1)^{n+1} \sin(\frac{1}{\sqrt{n}})$ converges by the AST.

However, since the FTL shows that

$$\lim_{n \to \infty} \frac{\sin(\frac{1}{\sqrt{n}})}{\frac{1}{\sqrt{n}}} = 1,$$

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, $\sum_{n=1}^{\infty} |(-1)^{n+1} \sin(\frac{1}{\sqrt{n}})| = \sum_{n=1}^{\infty} \sin(\frac{1}{\sqrt{n}})$ diverges. Hence $\sum_{n=1}^{\infty} (-1)^{n+1} \sin(\frac{1}{\sqrt{n}})$ is not absolutely convergent.

Definition: [Conditional Convergence]

A series $\sum_{n=1}^{\infty} a_n$ is said to be *conditionally convergent* if it converges, but it is not absolutely convergent.

Question 4: Why do we care if a series converges absolutely rather than conditionally?

Question 5: For a finite sum order does not matter. That is

$$a+b+c+d = d+c+b+a$$

Is this true for infinite sums?

Definition: [Rearrangement of a Series] Given a sequence $\{a_n\}$ and a 1-1 and onto function $\phi : \mathbb{N} \to \mathbb{N}$, if we let

 $b_n = a_{\phi(n)},$

then the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_{\phi(n)}$ is called a *rearrangement* of the series $\sum_{n=1}^{\infty} a_n$.

Question 6: What do we know about the convergence of $\sum_{n=1}^{\infty} a_{\phi(n)}$ relative to that of $\sum_{n=1}^{\infty} a_n$. In particular, if $\sum_{n=1}^{\infty} a_n$ converges, does $\sum_{n=1}^{\infty} a_{\phi(n)}$ also converge with

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}?$$

Facts:

If ∑_{n=1}[∞] a_n converges absolutely, then ∑_{n=1}[∞] a_n = ∑_{n=1}[∞] a_{φ(n)} for every rearrangement ∑_{n=1}[∞] a_{φ(n)} of ∑_{n=1}[∞] a_n.
If ∑_{n=1}[∞] a_n converges conditionally, then ∑_{n=1}[∞] a_{φ(n)} may or may not converge.
If ∑ a_n converges conditionally and if α ∈ ℝ ∪ {-∞, ∞}, then there

exists a 1-1 and onto function $\phi:\mathbb{N}\to\mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\phi(n)} = \alpha.$$

Summary: Absolutely convergent series are stable, conditionally convergent series are not!