# Alternating Series Test Part II: The AST and Error Estimation

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# Alternating Series

**Recall:** If $a_n = \frac{1}{n}$, then

i) $a_n > 0$

ii) $a_{n+1} < a_n$

iii) $\lim_{n\to\infty} a_n = 0$

Using these properties we showed that the *Alternating Series*

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1}\frac{1}{n} + \cdots$$

converged. Moreover, if

$$S_k = \sum_{n=1}^{k} (-1)^{n+1}\frac{1}{n} \quad \text{and} \quad S = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n},$$

then

$$|S - S_k| < a_{k+1} = \frac{1}{k+1}$$

# Alternating Series

**Theorem: [Alternate Series Test]**

1) $a_n > 0$ for all $n \in \mathbb{N}$,
2) $a_{n+1} < a_n$ for all $n \in \mathbb{N}$,
3) $\lim_{n\to\infty} a_n = 0$.

Then the series

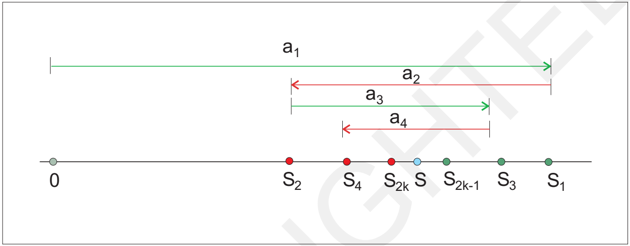
$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Moreover, if $S_k = \sum_{n=1}^{k} (-1)^{n+1} a_n$ and $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$, then

$$\mid S_k - S \mid < a_{k+1}.$$

# Alternating Series




We get

1) $\{S_{2k-1}\}$ is decreasing and bounded below by $0$
   $\Rightarrow \{S_{2k-1}\} \to L$.
2) $\{S_{2k}\}$ is increasing and bounded above by $a_1 \Rightarrow \{S_{2k}\} \to M$.
3) $\lim\limits_{k\to\infty} \mid S_{2k-1} - S_{2k} \mid = \lim\limits_{k\to\infty} a_{2k} = 0 \Rightarrow L = M = S.$

Finally, since $S$ is between $S_k$ and $S_{k+1}$ for each $k$,

$$\mid S_k - S \mid < \mid S_k - S_{k+1} \mid = a_{k+1}.$$

# Example:

**Example:** Show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5}$ converges to some $S$ and that

$$\left|\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5} - \frac{31}{32}\right| < \frac{1}{100}.$$

**Solution:** With $a_n = \frac{1}{n^5}$ it is clear that

1) $a_n > 0$ for all $n \in \mathbb{N}$,
2) $a_{n+1} < a_n$ for all $n \in \mathbb{N}$,
3) $\lim_{n\to\infty} a_n = 0$.

so the series converges by the AST.

We know that $\frac{31}{32} = 1 - \frac{1}{2^5} = \sum_{n=1}^{2}(-1)^{n+1}\frac{1}{n^5} = S_2$. Hence

$$\left|\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5} - \frac{31}{32}\right| = |S - S_2| < a_3 = \frac{1}{3^5} = \frac{1}{243} < \frac{1}{100}.$$

# Example:

**Example:** The Integral Test shows that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

converges and the AST shows that

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln(n))^2}$$

converges.
Let

$$T_k = \sum_{n=2}^{k} \frac{1}{n(\ln(n))^2} \quad \text{and} \quad T = \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

and let

$$S_k = \sum_{n=2}^{k} (-1)^n \frac{1}{n(\ln(n))^2} \quad \text{and} \quad S = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln(n))^2}$$

How large must $k$ be so that

$$|T - T_k| < \frac{1}{100} \quad \text{and} \quad |S - S_k| < \frac{1}{100}$$

respectively?

# Example:

**Solution:** The Integral Test shows that

$$|T - T_k| < \int_k^\infty \frac{1}{x(\ln(x))^2}\, dx = \frac{1}{\ln(k)}$$

We want

$$\frac{1}{\ln(k)} < \frac{1}{100} \Rightarrow k > e^{100} \cong 10^{43}$$

The AST shows that

$$|S - S_k| < \frac{1}{(k+1)(\ln(k+1))^2}$$

We want

$$\frac{1}{(k+1)(\ln(k+1))^2} < \frac{1}{100} \Rightarrow k \geq 14$$