Alternating Series Test Part I: Introduction

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Alternating Series

Recall: The *Harmonic Series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

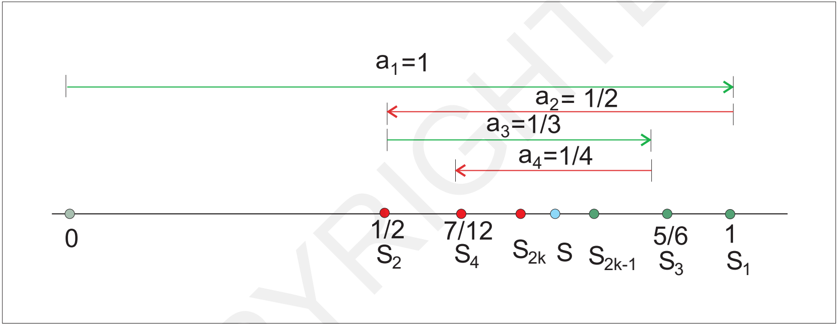
diverges.

Question: What can we say about the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1} \frac{1}{n} + \cdots ?$$

This series is called the *Alternating Series*.

Alternating Series



Let $a_n = \frac{1}{n}$ and $S_k = \sum_{n=1}^{k} (-1)^{n+1} \frac{1}{n}$.

1) $\{S_{2k-1}\}$ is decreasing and bounded below.

2) $\{S_{2k}\}$ is increasing and bounded above.

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Claim 1: $\{S_{2k-1}\}$ is decreasing and bounded below by 0.

1) $S_{2k+1} - S_{2k-1} = \sum_{n=1}^{2k+1} (-1)^{n+1}\frac{1}{n} - \sum_{n=1}^{2k-1} (-1)^{n+1}\frac{1}{n} =$
$-\frac{1}{2k} + \frac{1}{2k+1} < 0$ so $\{S_{2k-1}\}$ is decreasing.
2) $S_{2k-1} = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{2k-3} - \frac{1}{2k-2}) + \frac{1}{2k-1} > 0.$

Conclusion: The MCT shows that $\{S_{2k-1}\}$ converges.

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Claim 2: $\{S_{2k}\}$ is increasing and bounded above by 1.

1) $S_{2k+2} - S_{2k} = \sum_{n=1}^{2k+2} (-1)^{n+1}\frac{1}{n} - \sum_{n=1}^{2k} (-1)^{n+1}\frac{1}{n} = \frac{1}{2k+1} - \frac{1}{2k+2} > 0$ so $\{S_{2k}\}$ is increasing.
2) $S_{2k} = 1 + (-\frac{1}{2}+\frac{1}{3}) + (-\frac{1}{4}+\frac{1}{5}) + \cdots + (-\frac{1}{2k-2}+\frac{1}{2k-1}) - \frac{1}{2k} < 1.$

Conclusion: The MCT shows that $\{S_{2k}\}$ converges.

Note:

$$\mid S_{2k} - S_{2k-1} \mid = \frac{1}{2k} \to 0.$$

Therefore,

$$\lim_{k\to\infty} S_{2k-1} = S = \lim_{k\to\infty} S_{2k}$$

and $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ converges (to $\ln(2)$).

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Theorem: [Alternate Series Test]

1) $a_n > 0$ for all $n \in \mathbb{N}$,
2) $a_{n+1} < a_n$ for all $n \in \mathbb{N}$,
3) $\lim_{n \to \infty} a_n = 0$.

Then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Moreover, if $S_k = \sum_{n=1}^{k} (-1)^{n+1} a_n$ and $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$, then

$$| S_k - S | < a_{k+1}.$$

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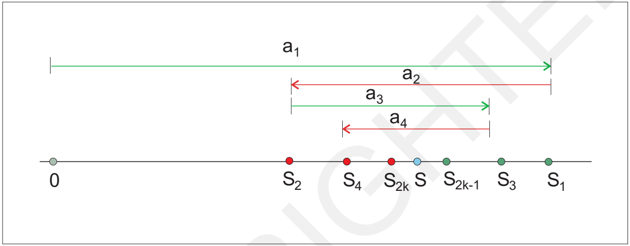

We get

1) $\{S_{2k-1}\}$ is decreasing and bounded below by 0
   $\Rightarrow \{S_{2k-1}\} \to L$.
2) $\{S_{2k}\}$ is increasing and bounded above by $a_1 \Rightarrow \{S_{2k}\} \to M$.
3) $\lim_{k\to\infty} \mid S_{2k-1} - S_{2k} \mid = \lim_{k\to\infty} a_{2k} = 0 \Rightarrow L = M = S.$

Finally, since S is between S_k and S_{k+1} for each k,

$$\mid S_k - S \mid < \mid S_k - S_{k+1} \mid = a_{k+1}.$$

Example:

Example: Show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5}$ converges to some S and that

$$\left|\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5} - \frac{31}{32}\right| < \frac{1}{100}.$$

Solution: With $a_n = \frac{1}{n^5}$ it is clear that

1) $a_n > 0$ for all $n \in \mathbb{N}$,
2) $a_{n+1} < a_n$ for all $n \in \mathbb{N}$,
3) $\lim_{n\to\infty} a_n = 0$.

so the series converges by the AST.

We know that $\frac{31}{32} = 1 - \frac{1}{2^5} = \sum_{n=1}^{2}(-1)^{n+1}\frac{1}{n^5} = S_2$. Hence

$$\left|\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5} - \frac{31}{32}\right| = |S - S_2| < a_3 = \frac{1}{3^5} = \frac{1}{243} < \frac{1}{100}.$$