Separable Differential Equations

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Separable Differential Equations

Question: How do we solve the differential equation

$$y' = x(y^2 + 1)?$$

Note: We saw that the function

$$y = \tan(\frac{x^2}{2})$$

satisfies the above equation but how did we discover this?

We can divide by $y^2 + 1$ to get

$$\frac{y'}{y^2 + 1} = x$$

Integrating both sides with respect to x gives

$$\int \frac{y'}{y^2 + 1}\, dx = \int x\, dx$$

Separable Differential Equations

Question Cont'd: How do we solve the differential equation

$$y' = x(y^2 + 1)?$$

Since $y = y(x)$, we can apply the Change of Variables theorem get that

$$\begin{aligned} \int \frac{y'}{y^2+1}\,dx &= \int \frac{1}{(y(x))^2+1} y'(x)\,dx \\ &= \int \frac{1}{y^2+1} dy \\ &= \arctan(y) \end{aligned}$$

Hence

$$\arctan(y) = \int x\,dx = \frac{x^2}{2} + C$$

Solving for y in terms of x gives us

$$y = \tan(\frac{x^2}{2} + C)$$

Separable Differential Equations

Definition: [Separable Differential Equation]

We say that a first order differentiable equation is *separable* if there exists functions $f = f(x)$ and $g = g(y)$ such that the equation can be written in the form

$$y' = f(x)g(y)$$

Example:

1. $y' = x(y^2 + 1)$ is separable. In this case,

$$f(x) = x \quad \text{and} \quad g(y) = y^2 + 1$$

2. $y' = y$ is separable. In this case,

$$f(x) = 1 \quad \text{and} \quad g(y) = y$$

3. $y' = \cos(xy)$ is not separable.

Solving Separable Differential Equations

Question: How do we solve a separable differential equation

$$y' = f(x)g(y)$$

Step 1: Finding all constant solutions.

Suppose that $y_0 \in \mathbb{R}$ is such that

$$g(y_0) = 0$$

Let

$$\phi(x) = y_0$$

for all $x \in \mathbb{R}$. Then

$$\phi'(x) = 0 = f(x)g(y_0) = f(x)g(\phi(x))$$

for each $x \in \mathbb{R}$.

Solving Separable Differential Equations

Definition: [Constant Solutions to Separable DE's]

If

$$y' = f(x)g(y)$$

is a separable differential equation and if $y_0 \in \mathbb{R}$ is such that $g(y_0) = 0$, then

$$\phi(x) = y_0$$

is called a *constant or equilibrium solution* to the differential equation.

Solving Separable Differential Equations

Example: Find all constant solutions to the equation

$$y' = y(1 - y)$$

Solution : We have

$$f(x) = 1 \quad \text{and} \quad g(y) = y(1 - y)$$

Therefore, $g(y_0) = 0$ if

$$y_0 = 0 \quad \text{or} \quad y_0 = 1$$

The constant solutions are

$$\varphi(x) = 0 \quad \text{or} \quad \psi(x) = 1$$

for all $x \in \mathbb{R}$.

Note: The separable differential equation

$$y' = x(y^2 + 1)$$

has no constant solutions since $y^2 + 1 > 0$ for all $y \in \mathbb{R}$.

Solving Separable Differential Equations

Step 2: Finding all implicit solutions.

When $g(y) \neq 0$, we can divide by $g(y)$ to get

$$\frac{y'}{g(y)} = f(x)$$

Integrating both sides with respect to x gives

$$\int \frac{y'}{g(y)}\,dx = \int f(x)\,dx$$

Since $y = y(x)$, the Change of Variables Theorem gives

$$\begin{aligned}\int \frac{y'}{g(y)}\,dx &= \int \frac{1}{g(y(x))} y'(x)\,dx \\ &= \int \frac{1}{g(y)}\,dy\end{aligned}$$

Hence

$$G(y) = \int \frac{1}{g(y)}\,dy = \int f(x)\,dx = F(x) + C$$

The expression

$$G(y) = F(x) + C$$

is called an *implicit solution* to our equation.

Solving Separable Differential Equations

Step 3: Finding all explicit solutions.

Given the equation

$$G(y) = \int \frac{1}{g(y)} dy = \int f(x)\, dx = F(x) + C$$

our last step is to try and solve the implicit equation

$$G(y) = F(x) + C$$

for y in terms of x to get our explicit solutions.

Solving Separable Differential Equations

Example: Find all solutions to the equation

$$y' = (x+1)y$$

Step 1: Since $g(y) = y$, the only y_0 such that $g(y_0) = 0$ is $y_0 = 0$. Therefore,

$$y(x) = 0$$

is the only constant solution.

Step 2: If $y \neq 0$, we have

$$\int \frac{1}{y}\, dy = \int (x+1)\, dx$$

so

$$\ln(|\, y\,|) = \frac{x^2}{2} + x + C$$

where C is an arbitrary constant.

Solving Separable Differential Equations

Step 3: Since

$$\ln(|\, y \,|) = \frac{x^2}{2} + x + C$$

we get

$$\begin{aligned} |\, y \,| &= e^{\ln(|y|)} \\ &= e^{\frac{x^2}{2}+x+C} \\ &= e^C e^{\frac{x^2}{2}+x} \\ &= C_1 e^{\frac{x^2}{2}+x} \end{aligned}$$

where $C_1 = e^C > 0$.
However,

$$|\, y \,| = C_1 e^{\frac{x^2}{2}+x}$$

means that

$$\begin{aligned} y &= \pm C_1 e^{\frac{x^2}{2}+x} \\ &= C_2 e^{\frac{x^2}{2}+x} \end{aligned}$$

where $C_2 = \pm C_1 \neq 0$.

Solving Separable Differential Equations

Remark: The solutions to the equation

$$y' = (x+1)y$$

are either the constant solution

$$y(x) = 0$$

or the solutions of the form

$$y = Ce^{\frac{x^2}{2}+x}$$

where $C \neq 0$.
Observe that

$$y = 0 = 0 \cdot e^{\frac{x^2}{2}+x}$$

so in this example all solutions are of the form

$$y = Ce^{\frac{x^2}{2}+x}$$

where $C \in \mathbb{R}$.

Solving Separable Differential Equations

Remark: It may not be immediately obvious that an differential equation is separable. For example

$$y' = xy + y$$

does not immediately look separable.

However the equation can be rewritten as

$$y' = (x + 1)y$$

so it is separable with

$$f(x) = x + 1 \quad \text{and} \quad g(y) = y$$

Solving Separable Differential Equations

Solving the separable differential equation

$$y' = f(x)g(y)$$

Step 1: Determine whether the DE is separable. You may have to factor the DE to identify $f(x)$ and $g(y)$.

Step 2: Determine the *constant solution(s)* by finding all the values y_0 such that $g(y_0) = 0$. For each such y_0, the constant function

$$y = y(x) = y_0$$

is a solution.

Step 3: If $g(y) \neq 0$, integrate both sides of the following equation

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

to solve the differential equation implicitly.

Step 4: Solve the implicit equation from Step 3 explicitly for y in terms of x.

Step 5: [Optional] Check your solution by differentiating y to determine if this derivative does satisfy the original equation.