Logistic Growth

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Logistic Growth

Remark: A population with unlimited resources grows at a rate that is proportional to its size. That is

$$P' = kP$$

If there is a maximum population M that the resources can support then typically

$$P' = kP(M - P)$$

This population satisfies a *logistic growth* model and

$$y' = ky(M - y)$$

is called the *logistic equation*.

Solving the Logistic Equation

Note: The logistic equation

$$P' = kP(M-P)$$

is separable with constant solutions $P(t) = 0$ and $P(t) = M$.

We have

$$\int \frac{1}{P(M-P)}\,dP = \int k\,dt = kt + C_1$$

and to evaluate

$$\int \frac{1}{P(M-P)}\,dP$$

we use partial fractions.

We get constants A and B are such that

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

or

$$1 = A(M-P) + B(P).$$

Solving the Logistic Equation

With

$$1 = A(M - P) + B(P),$$

letting $P = 0$ gives

$$1 = A(M)$$

so

$$A = \frac{1}{M}.$$

Letting $P = M$, we get

$$1 = B(M)$$

and again

$$B = \frac{1}{M}.$$

Therefore

$$\frac{1}{P(M-P)} = \frac{1}{M}\left[\frac{1}{P} + \frac{1}{M-P}\right].$$

Solving the Logistic Equation

It follows that

$$\begin{aligned}\int \frac{1}{P(M-P)}\,dP &= \frac{1}{M}\left[\int \frac{1}{P}\,dP + \int \frac{1}{M-P}\,dP\right]\\ &= \frac{1}{M}[\ln(\mid P\mid) - \ln(\mid M-P\mid)] + C_2\\ &= \frac{1}{M}\ln\left(\frac{\mid P\mid}{\mid M-P\mid}\right) + C_2\end{aligned}$$

We now have that

$$\frac{1}{M}\ln\left(\frac{\mid P(t)\mid}{\mid M-P(t)\mid}\right) + C_2 = kt + C_1.$$

Therefore,

$$\ln\left(\frac{\mid P(t)\mid}{\mid M-P(t)\mid}\right) = Mkt + C_3$$

where C_3 is arbitrary.

This shows that

$$\frac{\mid P(t)\mid}{\mid M-P(t)\mid} = Ce^{Mkt}$$

where $C = e^{C_3} > 0$.

Solving the Logistic Equation

Case 1: Assume that $0 < P(t) < M$. Then

$$\frac{|\, P(t)\, |}{|\, M - P(t)\, |} = \frac{P(t)}{M - P(t)} = Ce^{Mkt}.$$

Solving for $P(t)$ would give

$$\begin{aligned} P(t) &= (M - P(t))Ce^{Mkt} \\ &= MCe^{Mkt} - P(t)Ce^{Mkt} \end{aligned}$$

so that

$$P(t) + P(t)Ce^{Mkt} = MCe^{Mkt}.$$

We then have

$$P(t)(1 + Ce^{Mkt}) = MCe^{Mkt}$$

and finally that

$$\begin{aligned} P(t) &= \frac{MCe^{Mkt}}{1 + Ce^{Mkt}} \\ &= M\frac{Ce^{Mkt}}{1 + Ce^{Mkt}} \end{aligned}$$

Solving the Logistic Equation

Two Observations:

1) Since $C > 0$, the denominator is never 0 so the function $P(t)$ is continuous and

$$0 < \frac{Ce^{Mkt}}{1 + Ce^{Mkt}} < 1$$

so that

$$0 < P(t) < M$$

which agrees with our assumption.

2) Since $k > 0$, we have that

$$\begin{aligned}\lim_{t\to\infty} P(t) &= \lim_{t\to\infty} M\frac{Ce^{Mkt}}{1 + Ce^{Mkt}} \\ &= M\lim_{t\to\infty} \frac{Ce^{Mkt}}{1 + Ce^{Mkt}} \\ &= M\end{aligned}$$

and

$$\lim_{t\to-\infty} P(t) = \lim_{t\to-\infty} M\frac{Ce^{Mkt}}{1 + Ce^{Mkt}} = 0.$$

Solving the Logistic Equation

If $t = 0$, then

$$P_0 = P(0) = M\frac{Ce^0}{1+Ce^0} = M\frac{C}{1+C}.$$

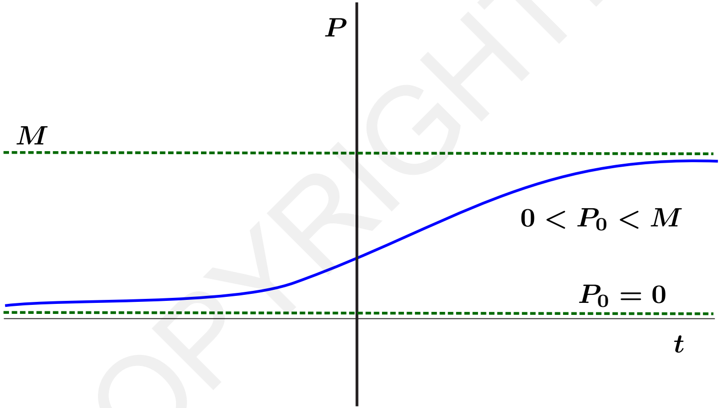
Solving for C yields

$$P_0(1+C) = MC$$
$$P_0 + P_0C = MC$$
$$P_0 = (M - P_0)C$$

and finally that

$$C = \frac{P_0}{M - P_0}.$$

Logistic Growth



Logistic Growth

Solving the Logistic Equation

Case 2: If $P(0) > M$, then

$$\frac{|\ P(t)\ |}{|\ M - P(t)\ |} = -\frac{P(t)}{M - P(t)} = \frac{P(t)}{P(t) - M} = Ce^{Mkt}.$$

We get that there exists a positive constant C such that

$$P(t) = M\frac{Ce^{Mkt}}{Ce^{Mkt} - 1}.$$

Note: This function has a vertical asymptote when the denominator

$$Ce^{Mkt} - 1 = 0.$$

Moreover, the function is only positive if

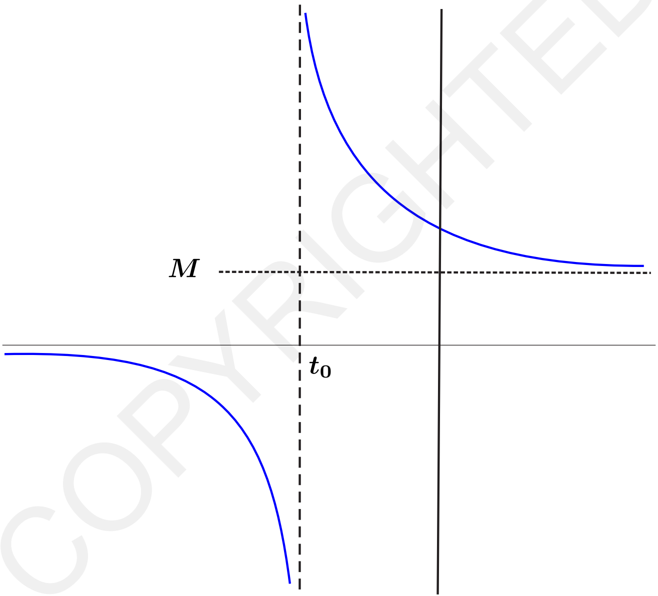
$$Ce^{Mkt} > 1$$

or equivalently if

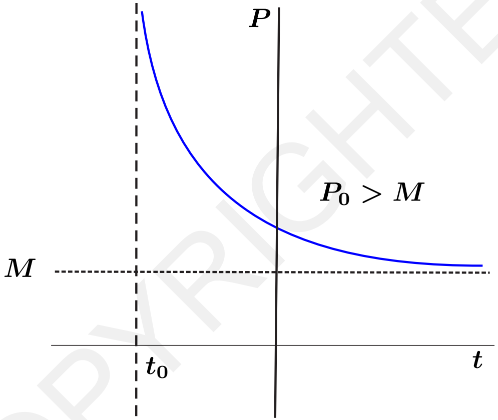
$$e^{Mkt} > \frac{1}{C}.$$

This happens if and only if $t > \frac{\ln\left(\frac{1}{C}\right)}{Mk} = t_0$.

Solving the Logistic Equation



Solving the Logistic Equation



Note: Since we are looking for a population function and so we require $P(t) \geq 0$, we will only consider values of t which exceed t_0. Therefore, the graph of the population function is as above.

Solving the Logistic Equation

Example: A game reserve can support at most 800 elephants. An initial population of 50 elephants is introduced in the park. After 5 years the population has grown to 120 elephants. Assuming that the population satisfies a logistic growth model, how large will the population be 25 years after this introduction?

Let $P(t)$ denote the elephant population t years after they are introduced to the park. There are positive constants C and k such that the population of elephants is given by

$$P(t) = 800\frac{Ce^{800kt}}{1 + Ce^{800kt}}.$$

If $P_0 = P(0)$, then

$$C = \frac{P_0}{M - P_0}.$$

Since $P_0 = P(0) = 50$ and $M = 800$. Then

$$C = \frac{50}{800 - 50} = \frac{50}{750} = \frac{1}{15}.$$

Therefore,

$$P(t) = 800\frac{\frac{1}{15}e^{800kt}}{1 + \frac{1}{15}e^{800kt}}.$$

Solving the Logistic Equation

Example Cont'd: To find k, we note that

$$120 = P(5) = 800\frac{\frac{1}{15}e^{800k(5)}}{1+\frac{1}{15}e^{800k(5)}}.$$

Hence

$$\frac{120}{800} = \frac{3}{20} = \frac{\frac{1}{15}e^{800k(5)}}{1+\frac{1}{15}e^{800k(5)}}$$

and thus

$$\frac{9}{4} = \frac{e^{4000k}}{1+\frac{1}{15}e^{4000k}}.$$

We get

$$\frac{9}{4}(1+\frac{1}{15}e^{4000k}) = e^{4000k}$$

so

$$\frac{9}{4} = \frac{17}{20}e^{4000k}.$$

This means

$$\frac{45}{17} = e^{4000k}$$

and finally that

$$k = \frac{\ln\left(\frac{45}{17}\right)}{4000}.$$

Solving the Logistic Equation

Example Cont'd: Substituting k back into the population model and evaluating at $t = 25$ we get

$$\begin{aligned} P(25) &= 800 \frac{\frac{1}{15} e^{800 \frac{\ln\left(\frac{45}{17}\right)}{4000}(25)}}{1 + \frac{1}{15} e^{800 \frac{\ln\left(\frac{45}{17}\right)}{4000}(25)}} \\ &= 800 \frac{\frac{1}{15} e^{5 \ln\left(\frac{45}{17}\right)}}{1 + \frac{1}{15} e^{5 \ln\left(\frac{45}{17}\right)}} \\ &= 717 \text{ elephants} \end{aligned}$$