Linear Differential Equations

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Linear Differential Equations

Definition: [Linear Differential Equation]

A first order differential equation is said to be *linear* if it can be written in the form

$$y' = f(x)y + g(x)$$

Examples:

1. The separable differential equation

$$y' = 3x(y-1)$$

may be rewritten as

$$y' = 3xy - 3x$$

so it is also linear.

2. The differentiable equation

$$y' = x^2y^3$$

is not linear since the term y^3 is of third degree.

Linear Differential Equations

Question: How do we solve the linear differential equation

$$y' = f(x)y + g(x)?$$

Solving Linear Differential Equations

The Algorithm: Solve the linear differential equation

$$y' = 3xy - 3x$$

Step 1: Rewrite the equation as

$$y' - 3xy = -3x \quad (*)$$

Step 2: The Integrating Factor

Let $I = I(x)$ be a non-zero function and multiply each side of our DE by I to get

$$Iy' - 3xIy = -3xI$$

We also want $I = I(x)$ to satisfy

$$\frac{d}{dx}(Iy) = Iy' - 3xIy$$

From the product Rule we would need

$$Iy' + I'y = Iy' - 3xIy \quad (**)$$

and hence

$$I' = -3xI$$

Solving Linear Differential Equations

Step 2: The Integrating Factor Cont'd

The equation

$$I' = -3xI$$

is separable so we get

$$\ln(|I|) = \int \frac{1}{I}\, dI = \int -3x\, dx$$

We can choose any function of the form

$$I = e^{\int -3x\, dx} \Rightarrow I(x) = e^{-\frac{3x^2}{2}}$$

and we get from (**) that

$$\frac{d}{dx}(Iy) = Iy' + I'y = Iy' - 3xIy = -3xI(x)$$

This means that

$$Iy = \int \frac{d}{dx}(Iy)\, dx = \int -3xI(x)\, dx = \int -3xe^{-\frac{3x^2}{2}}\, dx \quad (***)$$

Solving Linear Differential Equations

Let $u = -\frac{3x^2}{2}$ to get $du = -3x dx$ and

$$\begin{aligned} \int -3xe^{-\frac{3x^2}{2}}\, dx &= \int e^u du \\ &= e^u + C \\ &= e^{-\frac{3x^2}{2}} + C \end{aligned}$$

Finally since

$$Iy = \int -3xe^{-\frac{3x^2}{2}}\, dx \quad (***)$$

we get

$$y = \frac{e^{-\frac{3x^2}{2}} + C}{I(x)} = \frac{e^{-\frac{3x^2}{2}} + C}{e^{-\frac{3x^2}{2}}} = 1 + Ce^{\frac{3x^2}{2}}$$

Solving Linear Differential Equations

Note: The function $I = I(x)$ in the previous example is called an *integrating factor*.

Solving Linear Differential Equations

Solving a First Order Linear DE

Step 1: Determine whether the DE is linear. Write the equation in the form

$$y' - f(x)y = g(x)$$

and identify $f(x)$ and $g(x)$.

Step 2: Calculate the integrating factor $I(x)$ with $I(x) \neq 0$. Solve for I by using

$$I = e^{-\int f(x)\,dx}$$

Step 3: Since $I(x) \neq 0$, the solution is

$$y = \frac{\int g(x)I(x)\,dx}{I(x)}$$

Step 4: [Optional] Check your solution by differentiating y.

Solving Linear Differential Equations

Theorem: [Solving First Order Linear Differential Equations]

Let f and g be continuous and let

$$y' = f(x)y + g(x)$$

be a first order linear differential equation. Then the solutions to this equation are of the form

$$y = \frac{\int g(x)I(x)\,dx}{I(x)}$$

where $I(x) = e^{-\int f(x)\,dx}$.

Solving Linear Differential Equations

Example : Solve the linear differential equation

$$y' = x - y$$

Solution : Since

$$f(x) = -1 \quad \text{and} \quad g(x) = x$$

the integrating factor is

$$I(x) = e^{-\int(-1)\,dx} = e^{x}$$

Therefore

$$y = \frac{\int g(x)I(x)\,dx}{I(x)} = \frac{\int xe^{x}\,dx}{e^{x}}$$

Integration by parts shows that

$$\int xe^{x}\,dx = xe^{x} - e^{x} + C$$

so

$$y = \frac{xe^{x} - e^{x} + C}{e^{x}} = x - 1 + Ce^{-x}$$