Introduction to Differential Equations

Created by

Barbara Forrest and Brian Forrest

Differential Equations

Example: Find all differentiable functions f satisfying the equation

$$f' = f \qquad (*)$$

Solution : It is easy to see that $f(x) = 0$ and $f(x) = e^x$ are *solutions* to the above equation.

As an application of Mean Value Theorem it can be shown that if f is a differentiable function satisfying the equation $(*)$, then there is a constant $C \in \mathbb{R}$ with

$$f(x) = Ce^x$$

for all $x \in \mathbb{R}$.

Differential Equations

Definition: [Differential Equation]

A *differential equation* is an equation involving an independent variable such as x, a function $y = y(x)$ and various derivatives of y.

We will often write

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0.$$

A *solution* to the differential equation is a function φ such that

$$F(x, \varphi(x), \varphi'(x), \cdots, \varphi^{(n)}(x)) = 0.$$

The highest order of a derivative appearing in the equation is called the *order* of the differential equation.

Differential Equations

Example: Consider the differential equation

$$F(x, y, y'') = (\cos(x))y + y'' = 0$$

This is an example of a differential equation of order 2.

The constant function $\varphi(x) = 0$ can easily be seen to be a solution to this equation since

$$\cos(x)\varphi + \varphi'' = \cos(x) \cdot 0 + 0 = 0$$

However at this point we have no tools to find any other solutions should they exist.

Differential Equations and Antiderivatives

Note:

1) In this course, we will typically consider only first order equations. Such DE's can be written in the form

$$y' = f(x, y)$$

A solution for a first order differentiable equation is a function φ for which

$$\varphi'(x) = f(x, \varphi(x))$$

2) The simplest first order DE is the equation

$$y' = f(x)$$

Hence $y = y(x)$ is a solution if and only if y is an antiderivative of f. Therefore, the solutions to this equation are given by

$$\int f(x)\, dx = F(x) + C$$

where F is any antiderivative of f and $C \in \mathbb{R}$ is an arbitrary constant.

In particular, each different choice of C results in a new solution.

The constant C is called a *parameter* and the collection $\{F(x) + C \mid C \in \mathbb{R}\}$ is called a *one parameter family.*

Differential Equations

Example: Solve the differential equation

$$y' = cos(x)$$

Solution: The set of solutions is given by

$$y = \int \cos(x)\, dx = \sin(x) + C$$

where $C \in \mathbb{R}$ is an arbitrary constant.

Differential Equations

Question: How do we solve the differential equation

$$y' = x(y^2 + 1)?$$

Note: The function

$$y = \tan(\frac{x^2}{2})$$

satisfies

$$\begin{aligned} y' &= x \cdot \sec^2(\frac{x^2}{2}) \\ &= x \cdot [\tan^2(\frac{x^2}{2}) + 1] \\ &= x(y^2 + 1) \end{aligned}$$

However

$$y = \tan(\frac{x^2}{2}) + 5$$

is not a solution to the above equation.