Initial Value Problems

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Initial Value Problems

Remark: Often a first order differential equation

$$y' = f(x, y)$$

will have infinitely many solutions. However, it is often the case that our solutions need to satisfy additional constraints.

Example: The population $P = P(t)$ of bacteria in a colony with unlimited resources satisfies the differential equation

$$P' = kP$$

The general solution is

$$P(t) = Ce^{kt}$$

Suppose that the population when we first counted the colony was P_0. So we can let

$$P(0) = P_0$$

Then

$$P_0 = P(0) = Ce^{k \cdot 0} = Ce^0 = C$$

and hence

$$P(t) = P_0 e^{kt}$$

The constraint that $P(0) = P_0$ is called an *initial value* or an *initial condition* .

Initial Value Problems

Initial Value Problem : Given a first order differential equation

$$y' = f(x, y)$$

and a set of initial values

$$\begin{aligned} y(x_0) &= y_0 \\ y(x_1) &= y_1 \\ y(x_2) &= y_2 \\ y(x_3) &= y_3 \\ &\vdots \end{aligned}$$

does there exist a solution to our differential equation which also satisfies the above constraints?

An Existence and Uniqueness Theorem

Theorem: [Existence and Uniqueness for First Order Linear DE's]

Assume that f and g are continuous functions on an interval I. Then for each $x_0 \in I$ and for all $y_0 \in \mathbb{R}$, the initial value problem

$$\begin{aligned} y' &= f(x)y + g(x) \\ y(x_0) &= y_0 \end{aligned}$$

has exactly *one* solution $y = \varphi(x)$ on the interval I.

Initial Value Problems

Example : Solve the initial value problem

$$y' = xy$$

with $y(0) = 1$.

Note : This equation is linear with integrating factor

$$I = e^{-\int x\,dx} = e^{-\frac{x^2}{2}}$$

and hence solutions of the form

$$y = \frac{\int 0 \cdot e^{-\frac{x^2}{2}}\,dx}{e^{-\frac{x^2}{2}}} = \frac{C}{e^{-\frac{x^2}{2}}} = Ce^{\frac{x^2}{2}}$$

Since

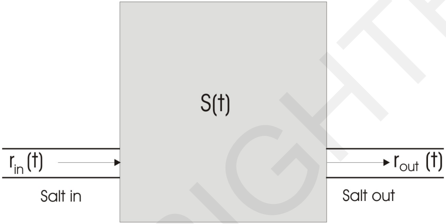
$$1 = y(0) = Ce^{\frac{0^2}{2}} = C$$

we get that

$$y = e^{\frac{x^2}{2}}$$

is the unique solution to our initial value problem.

A Mixing Problem

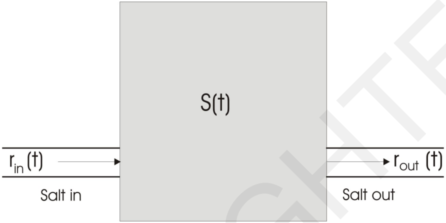


A Mixing Problem.

Assume that a brine containing $30g$ of salt per litre of water is pumped into a $1000\ l$ tank at a rate of 1 litre per second. The tank initially contains $1000\ l$ of fresh water. It also contains a device that thoroughly mixes its contents. The resulting solution is simultaneously drained from the tank at a rate of 1 litre per second.

Problem: How much salt will be in the tank at any given time?

A Mixing Problem



Let

$$
\begin{aligned}
s(t) &= \text{amount of salt in the tank at time } t \\
r_{\text{in}}(t) &= \text{rate at which salt enters the tank} \\
r_{\text{out}}(t) &= \text{rate at which salt leaves the tank}
\end{aligned}
$$

Then

$$
\begin{aligned}
r_{\text{in}}(t) &= 30g/l \times 1l/sec = 30g/sec \\
r_{\text{out}}(t) &= \frac{s(t)}{1000} g/l \times 1l/sec = \frac{s(t)}{1000} g/sec \\
s'(t) &= r_{\text{in}}(t) - r_{\text{out}}(t) = 30 - \frac{s(t)}{1000} g/sec
\end{aligned}
$$

A Mixing Problem

Note: We must solve the DE

$$s'(t) = 30 - \frac{s(t)}{1000}$$

with the initial condition that $s(0) = 0$.

This is a first order linear differential equation with integrating factor

$$I(t) = e^{-\int \frac{-1}{1000}\,dt} = e^{\frac{t}{1000}}$$

This gives us that

$$\begin{aligned} s(t) &= \frac{\int 30e^{\frac{t}{1000}}\,dt}{e^{\frac{t}{1000}}} \\ &= \frac{30000e^{\frac{t}{1000}} + C}{e^{\frac{t}{1000}}} \\ &= 30000 + Ce^{-\frac{t}{1000}} \end{aligned}$$

A Mixing Problem

Given that

$$s(t) = 30000 + Ce^{-\frac{t}{1000}}$$

grams of salt, if $s(0) = 0$, we get that

$$0 = 30000 + Ce^0 = 30000 + C$$

and hence

$$C = -30000$$

Therefore, at any given time

$$s(t) = 30000 - 30000e^{-\frac{t}{1000}}$$

grams.

Note: We have

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty} 30000 - 30000e^{-\frac{t}{1000}}g = 30000g$$

which is what you would expect in a $1000l$ tank with $30g/l$ of salt.