Graphical and Numerical Solutions of DE’s

Created by

Barbara Forrest and Brian Forrest

Graphical and Numerical Solutions of DE's

Remark: Most DE's can not be solved explicitly but we can use linear approximation to help us build both *graphical* and *numerical* approximations to solutions.

Recall: If f is differentiable at x_0, then f can be approximated near x_0 by its linear approximation

$$L_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$$

and the graph of L_{x_0} is the tangent line to the graph of f through the point $(x_0, f(x_0))$.

Direction Fields

Graphical Solution: The Strategy

- Given a differential equation

$$y' = f(x, y)$$

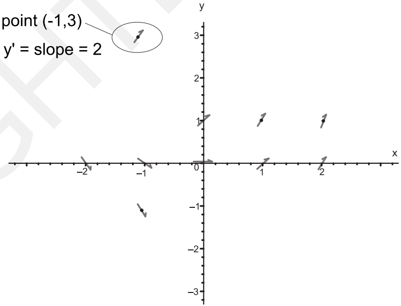
if there is a solution ϕ whose graph includes the point (x_0, y_0), then the slope of the tangent line to the graph of ϕ through (x_0, y_0) is

$$m = f(x_0, y_0)$$

- We can construct a local graphical approximation to the function ϕ near (x_0, y_0) by looking at a short segment of this tangent line.
- A collection of such segments is called a *direction field* for the differential equation.

Direction Fields

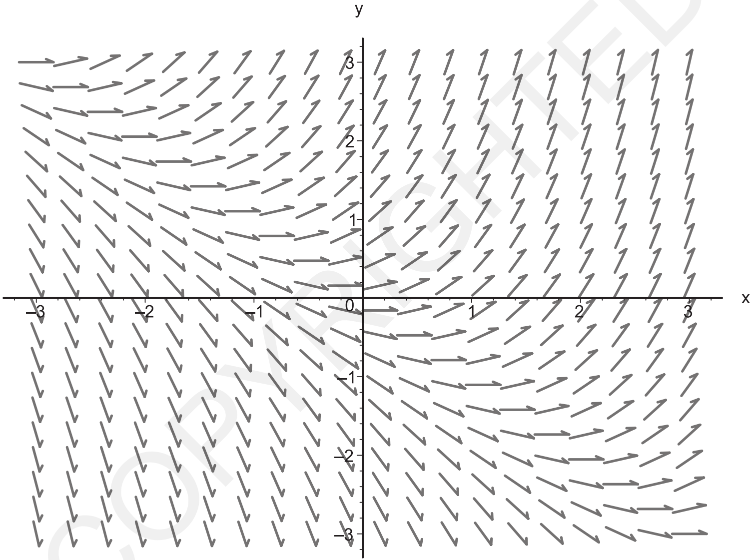
x	y	tangent line slope from DE $y' = x + y$
-2	0	$y' = -2 + 0 = -2$
-1	0	$y' = -1 + 0 = -1$
0	0	$y' = 0 + 0 = 0$
1	0	$y' = 1 + 0 = 1$
2	0	$y' = 2 + 0 = 2$
0	1	$y' = 0 + 1 = 1$
1	1	$y' = 1 + 1 = 2$
2	1	$y' = 2 + 1 = 3$
-1	3	$y' = -1 + 3 = 2$
-1	-1	$y' = -1 + -1 = -2$



Example: Consider the linear DE

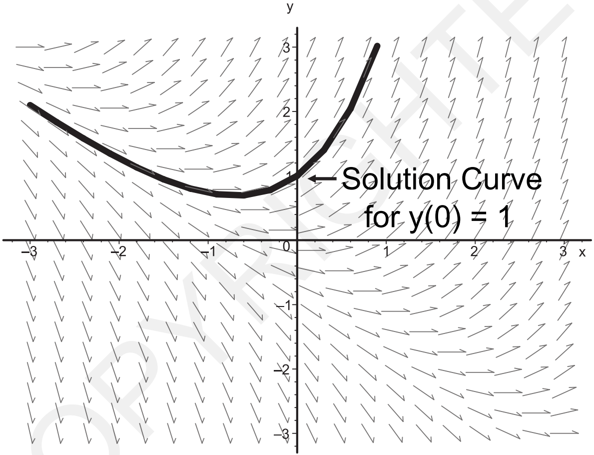
$$y' = x + y.$$

Direction Fields

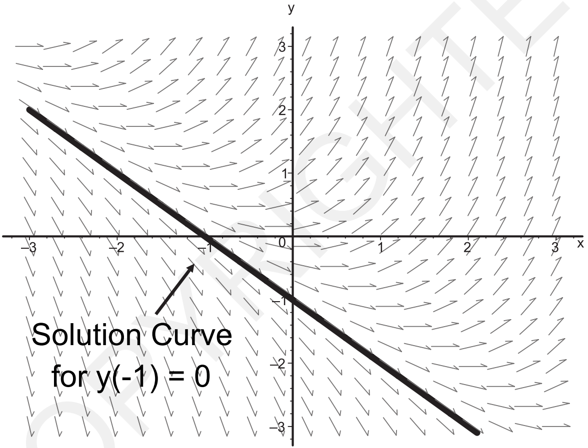


Direction Field for $y' = x + y$

Direction Fields



Direction Fields



Note: $y = -x - 1$ is a solution to $y' = x + y$ with $y(-1) = 0$.

Euler's Method

Euler's Method: This is an algorithm for building a numerical approximation on a closed interval $[a, b]$ to a solution to an initial value problem

$$y' = f(x, y)$$

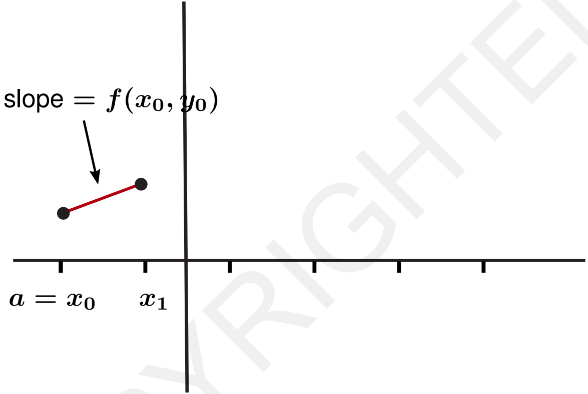
with $y(a) = y_0$.

Step 1: Determine a partition

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

of $[a, b]$

Euler's Method



Step 2: Assume that ϕ is a solution to $y' = f(x, y)$ with $\phi(x_0) = y_0$. Then

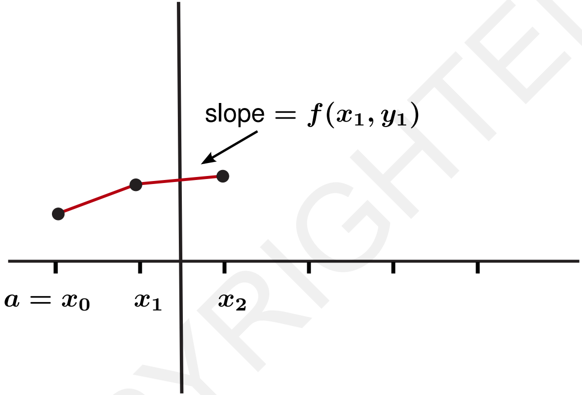
$$L_{x_0}(x) = y_0 + f(x_0, y_0)(x - x_0)$$

so we can assume that

$$\phi(x) = y_0 + f(x_0, y_0)(x - x_0)$$

on $[x_0, x_1]$.

Euler's Method



Step 3: Let

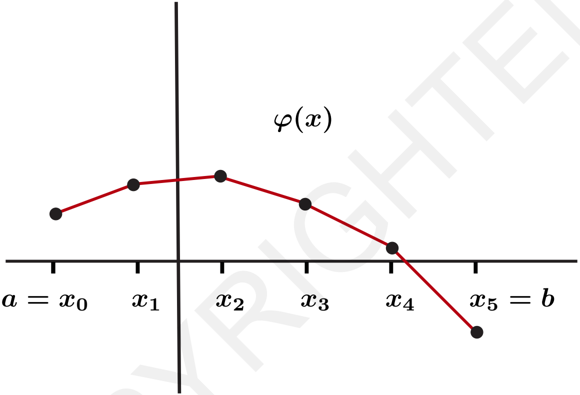
$$y_1 = L_{x_0}(x_1) = y_0 + f(x_0, y_0)(x_1 - x_0)$$

and define

$$\phi(x) = L_{x_1}(x) = y_1 + f(x_1, y_1)(x - x_1)$$

on $[x_1, x_2]$.

Euler's Method



Step 3: Let

$$y_2 = L_{x_1}(x_2) = y_1 + f(x_1, y_1)(x_2 - x_1)$$

and define

$$\phi(x) = L_{x_2}(x) = y_2 + f(x_2, y_2)(x - x_2)$$

on $[x_2, x_3]$ and continue until you reach $x_n = b$.