Volumes of Revolution:
Shell Method

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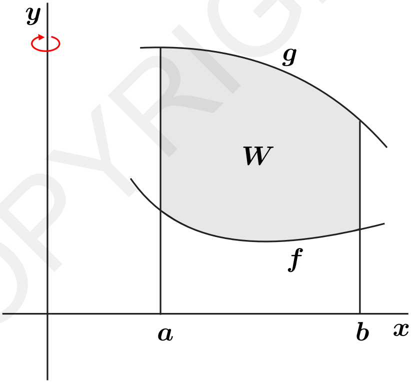
Volumes by the Shell Method

Problem: Assume that f and g are continuous on $[a, b]$, with $a \geq 0$ and $f(x) \leq g(x)$ on $[a, b]$.

Let W be the region bounded by the graphs of f and g and the lines $x = a$ and $x = b$.

Find the volume V of the solid obtained by rotating the region W around the y-axis.



Volumes by the Shell Method

Construct a regular n-partition of $[a, b]$ with

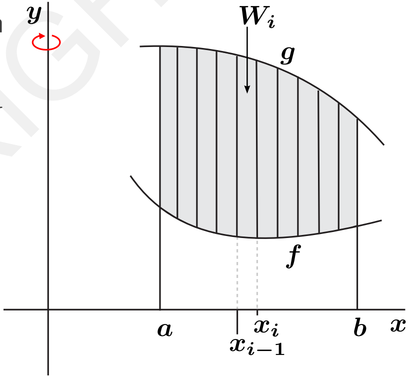
$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b.$$

This partition subdivides the region W into n subregions.

Let W_i denote the subregion of W on the interval $[x_{i-1}, x_i]$

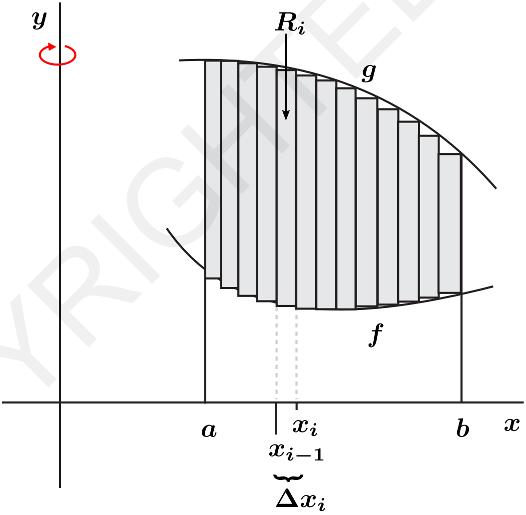
Let V_i be the volume obtained by rotating W_i around the y-axis so that

$$V = \sum_{i=1}^{n} V_i.$$



Volumes by the Shell Method

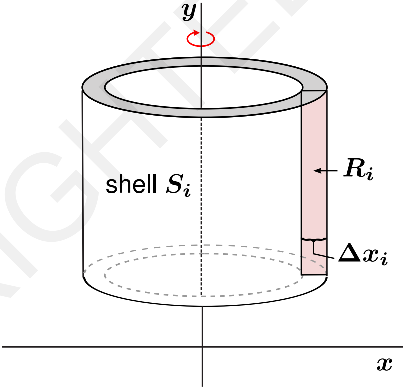
Approximate W_i by the rectangle R_i with height $g(x_i) - f(x_i)$, and base on the line $y = f(x_i)$ and top on the line $y = g(x_i)$ in the interval $[x_{i-1}, x_i]$.



Volumes by the Shell Method

If $\triangle x_i$ is small, then V_i is approximately equal to the volume obtained by rotating R_i around the y-axis.

Rotating R_i generates a thin cylindrical shell S_i.



For this reason, this method for finding volumes is called the *Shell Method* (or *Cylindrical Shell Method*).

Volumes by the Shell Method

The volume V_i^* of the shell generated by R_i is

$$\text{(circumference)} \times \text{(height)} \times \text{(thickness)}$$

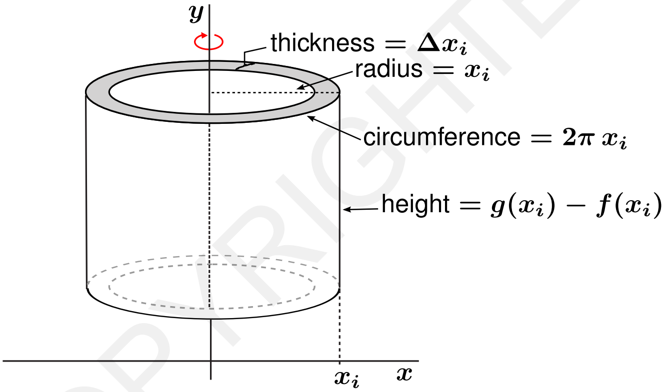
which is the same as

$$2\pi \times \text{(radius)} \times \text{(height)} \times \text{(thickness)}.$$

The height of the shell is $g(x_i) - f(x_i)$, its thickness is $\triangle x_i$, and the radius of revolution is x_i (the distance from the y-axis). Therefore, the volume V_i^* of S_i is

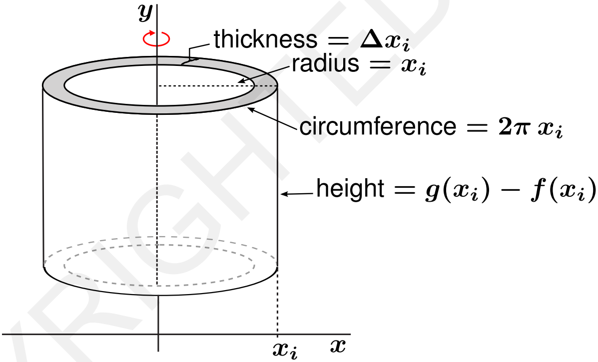
$$2\pi x_i(g(x_i) - f(x_i))\triangle x_i.$$

Volumes by the Shell Method



Volume $= 2\pi\, x_i\,(g(x_i) - f(x_i))\,\Delta x_i$

Volumes by the Shell Method



$$V = \sum_{i=1}^{n} V_i \cong \sum_{i=1}^{n} V_i^* = \sum_{i=1}^{n} 2\pi x_i (g(x_i) - f(x_i)) \triangle x_i$$

Let $n \to \infty$ to get

$$V = \int_a^b 2\pi x (g(x) - f(x))\, dx.$$

Volumes by the Shell Method

Volumes of Revolution: The Shell Method

Let $a \geq 0$. Let f and g be continuous on $[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$.

Let W be the region bounded by the graphs of f and g, and the lines $x = a$ and $x = b$.

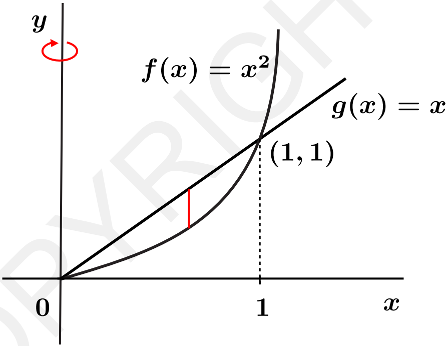
Then the volume V of the solid of revolution obtained by rotating the region W around the y-axis is given by

$$V = \int_a^b 2\pi x(g(x) - f(x))\, dx.$$

Volumes by the Shell Method

Example:

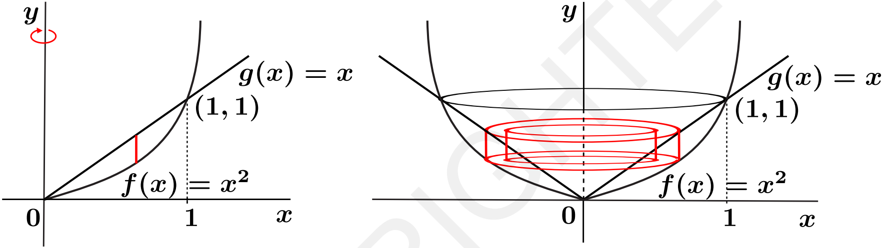
Find the volume of the solid obtained by revolving the closed region in the first quadrant bounded by the graphs of $g(x) = x$ and $f(x) = x^2$ around the y-axis.



You should verify that the graphs intersect in the first quadrant when $x = 0$ and $x = 1$ on the interval $[0, 1]$ with $f(x) \leq g(x)$.

Volumes by the Shell Method

Example (continued):



$$V = \int_0^1 2\pi x(g(x) - f(x))dx$$

$$= \int_0^1 2\pi x(x - x^2)dx$$

$$= \int_0^1 2\pi(x^2 - x^3)dx = \frac{\pi}{6}$$