Volumes of Revolution: Disk Method (Part 2)

Created by

Barbara Forrest and Brian Forrest

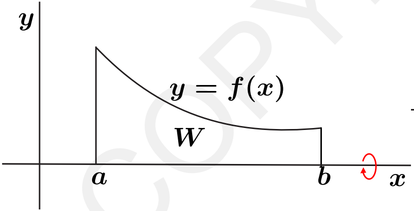
Volumes by the Disk Method (One Function)

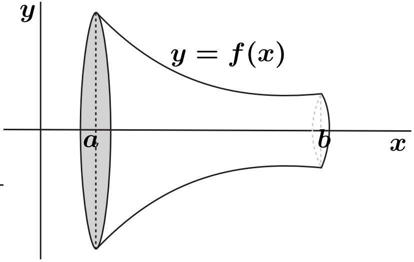
Recall:

Volumes of Revolution: The Disk Method I

Let f be continuous on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. Let W be the region bounded by the graphs of f, the x-axis and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by

$$V = \int_a^b \pi f(x)^2 dx.$$





Volumes by the Disk Method (General Case)

Until now we have looked at volume problems that involved a region bounded by a function f and the x-axis.

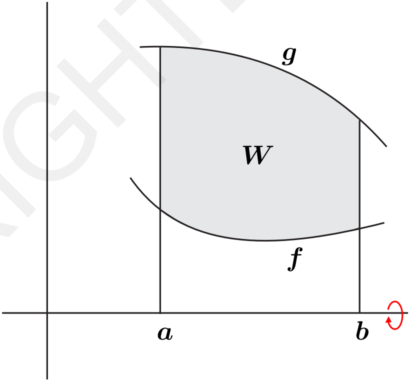
Now we will look at a more general problem where the region that is revolved is bounded by **two** functions.

Volumes by the Disk Method (General Case)

Problem:

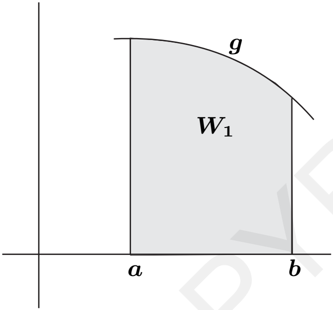
Suppose that $0 \leq f(x) \leq g(x)$.

We want to find the volume V of the solid formed by revolving the region W bounded by the graphs of f and g and the lines $x = a$ and $x = b$ around the x-axis.

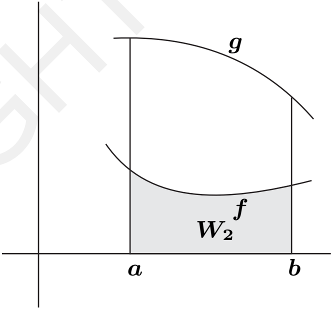


Volumes by the Disk Method (General Case)

Observe that if we let W_1 denote the region bounded by the graph of g, the x-axis, and the lines $x = a$ and $x = b$

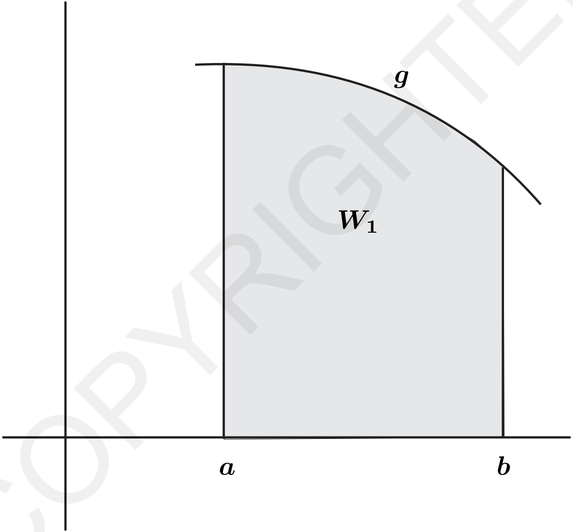


and we let W_2 denote region bounded by the graph f, the x-axis, and the lines $x = a$ and $x = b$,

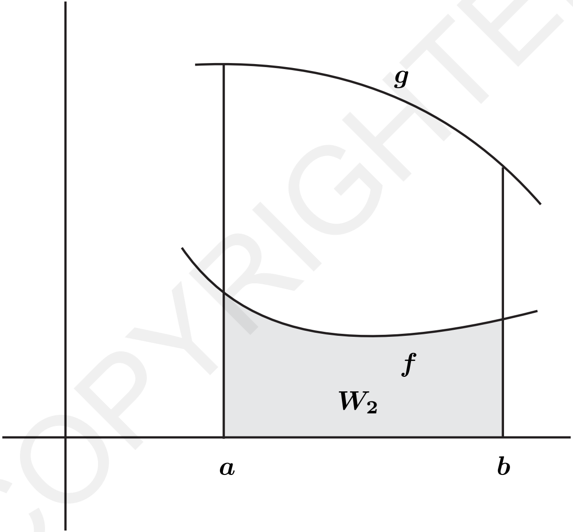


then W is the region that remains when we remove W_2 from W_1. It follows that the solid generated by revolving W around the x-axis is the same as the solid we would get by revolving W_1 around the x-axis and then removing the portion that would correspond to the solid obtained by revolving W_2 around the x-axis.

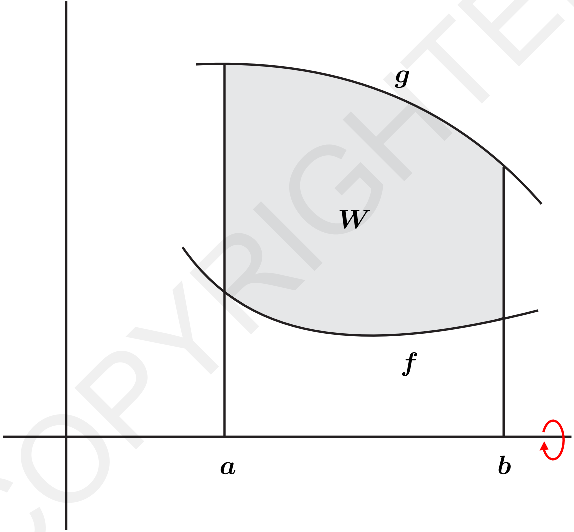
Volumes by the Disk Method (General Case)



Volumes by the Disk Method (General Case)



Volumes by the Disk Method (General Case)



Volumes by the Disk Method (General Case)

Let V_1 be the volume of the solid obtained by rotating W_1 and V_2 be the volume of the solid obtained by rotating W_2. Then

$$V = V_1 - V_2.$$

However, the formula for volumes of revolution tells us that

$$V_1 = \int_a^b \pi g(x)^2\,dx \qquad \text{and} \qquad V_2 = \int_a^b \pi f(x)^2\,dx.$$

Therefore,

$$\begin{aligned} V &= V_1 - V_2 \\ &= \int_a^b \pi g(x)^2\,dx - \int_a^b \pi f(x)^2\,dx \\ &= \int_a^b \pi (g(x)^2 - f(x)^2)\,dx \end{aligned}$$

Volumes by the Disk Method (General Case)

Volumes of Revolution: The Disk Method II

Let f and g be continuous on $[a, b]$ with $0 \leq f(x) \leq g(x)$ for all $x \in [a, b]$.

Let W be the region bounded by the graphs of f and g, and the lines $x = a$ and $x = b$.

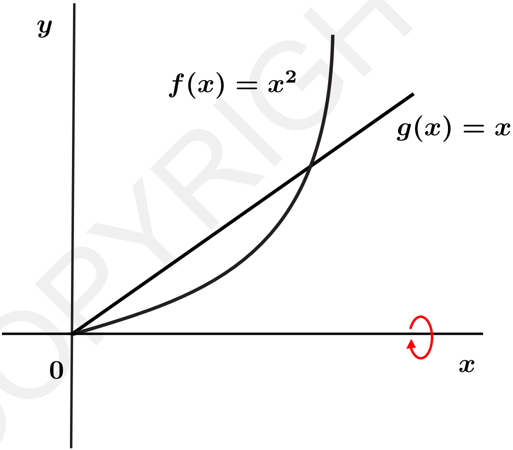
Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by

$$V = \int_a^b \pi(g(x)^2 - f(x)^2)\, dx.$$

Volumes by the Disk Method (General Case)

Example:

Find the volume V of the solid obtained by revolving the closed region bounded by the graphs of $g(x) = x$ and $f(x) = x^2$ around the x-axis.



Volumes by the Disk Method (General Case)

Example (continued):

Since we have not been given the interval over which we will integrate, we must find the x-coordinates of the points where the graphs intersect.

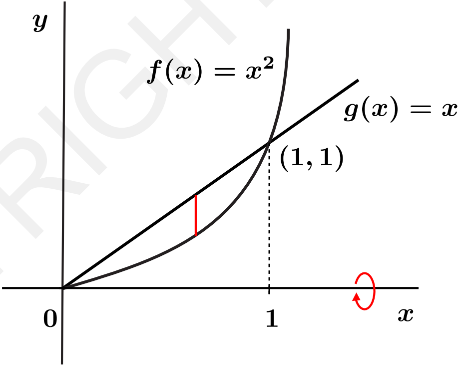
We must solve

$$x = x^2$$

so

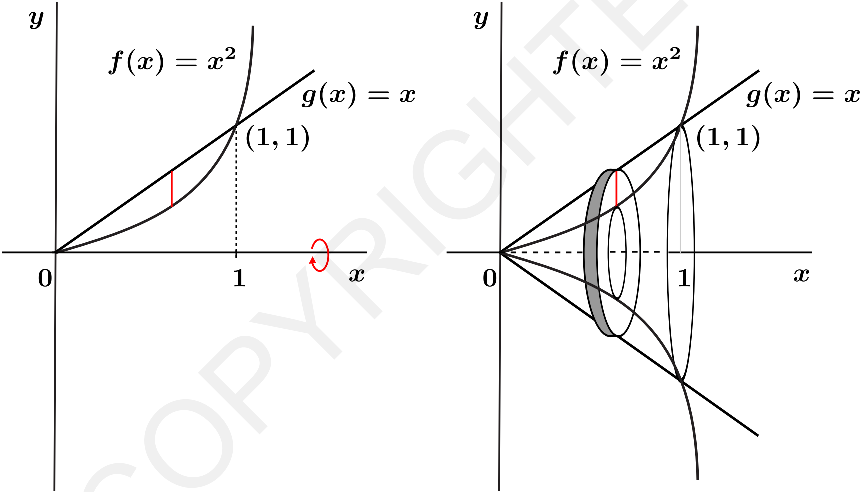
$$x^2 - x = x(x-1) = 0.$$

Hence, the points of intersection are located at $x = 0$ or $x = 1$.



Moreover, on $[0, 1]$, we have $0 \leq x^2 \leq x$.
(I.e., the graph of x^2 lies below the graph of x on this interval.)

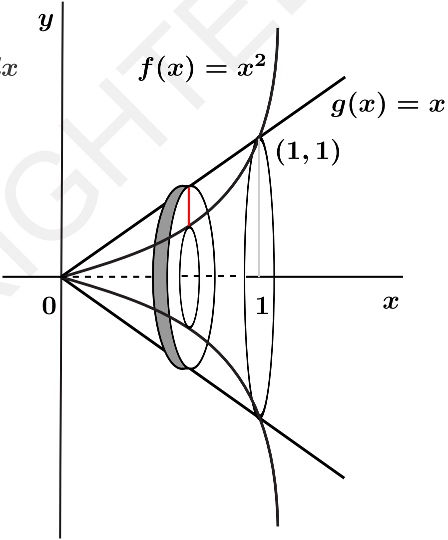
Volumes by the Disk Method (General Case)



Volumes by the Disk Method (General Case)

The volume is

$$
\begin{aligned}
V &= \int_0^1 \pi \left(g(x)^2 - f(x)^2\right)\, dx \\
&= \int_0^1 \pi \left((x)^2 - (x^2)^2\right)\, dx \\
&= \int_0^1 \pi \left(x^2 - x^4\right)\, dx \\
&= \pi \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 \\
&= \pi \left(\frac{1}{3} - \frac{1}{5}\right) \\
&= \frac{2\pi}{15}
\end{aligned}
$$

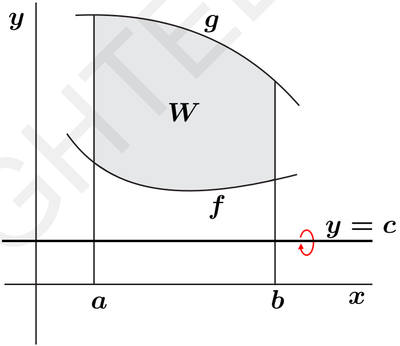


Volumes by the Disk Method (General Case)

Suppose that f and g are continuous on $[a, b]$ with

$$c \leq f(x) \leq g(x)$$

for all $x \in [a, b]$.



Let W be the region bounded by the graphs of f and g, and the lines $x = a$ and $x = b$.

What is the volume V of the solid of revolution obtained by revolving the region W around the line $y = c$?

The previous analysis still applies. Therefore, as an exercise, verify that the volume in this case is

$$V = \int_a^b \pi((g(x) - c)^2 - (f(x) - c)^2)\, dx.$$