Volumes of Revolution: Disk Method (Part 1)

Created by

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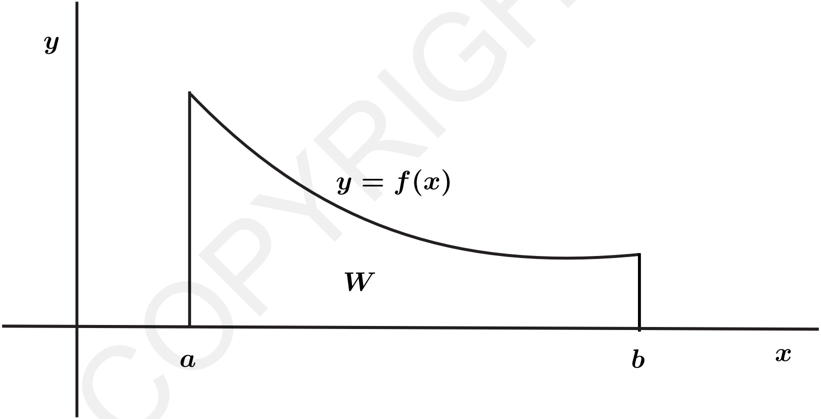
Volumes by the Disk Method

Problem:

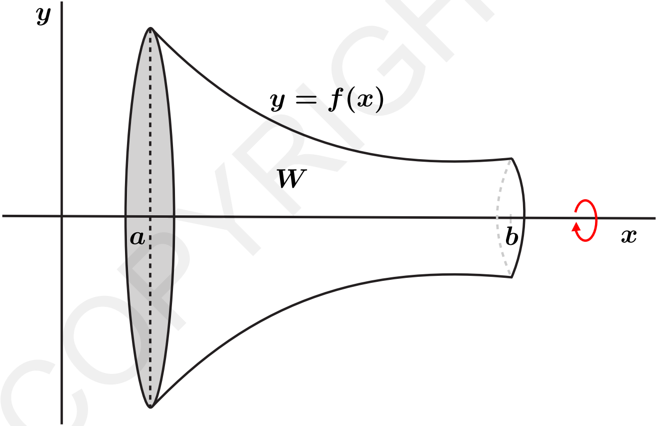
Assume that f is continuous on $[a, b]$ and that $f(x) \geq 0$ on $[a, b]$.

Let W be the region bounded by the graph of f, the lines $x = a$ and $x = b$ and the line $y = 0$.



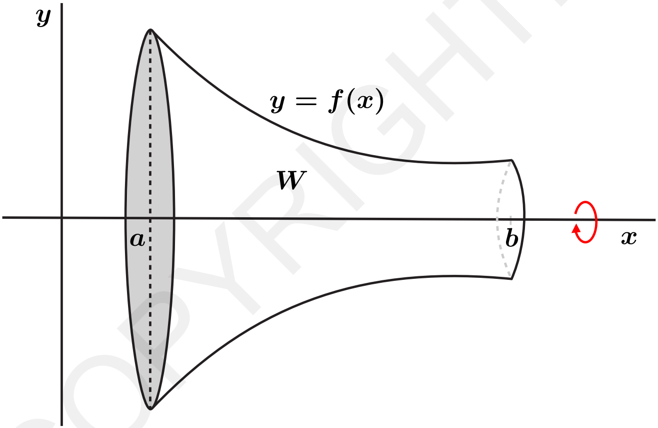
Volumes by the Disk Method

If region W is revolved around the x-axis an object called a *solid of revolution* is generated with the property that each vertical cross section of the solid is a circle with radius equal to the value of the function at the location of the slice.



Volumes by the Disk Method

Goal: Determine the volume V of this solid.



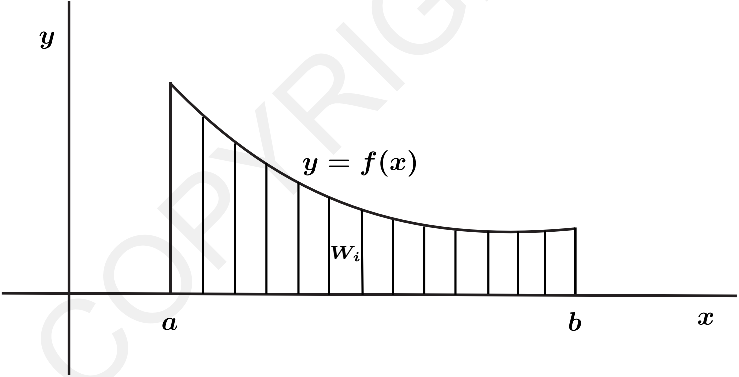
Volumes by the Disk Method

Using integration we begin with a regular n-partition

$$a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b$$

of $[a, b]$ with $\Delta t_i = \frac{b-a}{n}$ and $t_i = a + \frac{i(b-a)}{n}$.

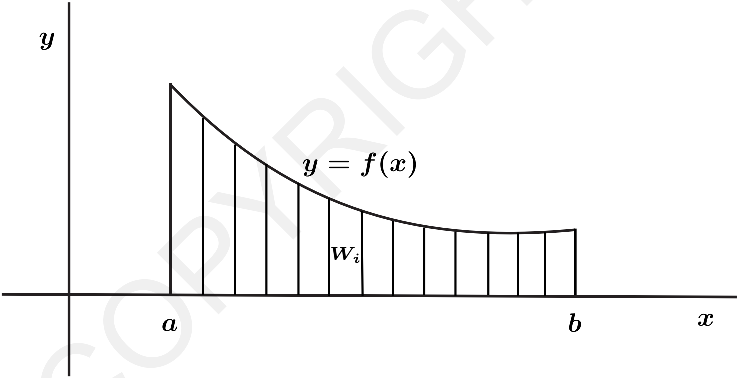
This partition subdivides the region W into n subregions. Let W_i denote the subregion of W in the interval $[x_{i-1}, x_i]$.



Volumes by the Disk Method

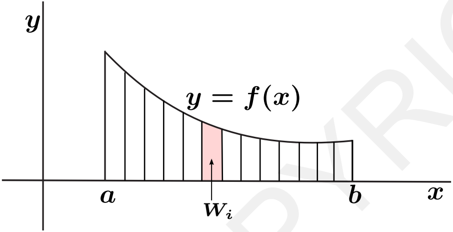
Let V_i be the volume obtained by rotating W_i around the axis, then

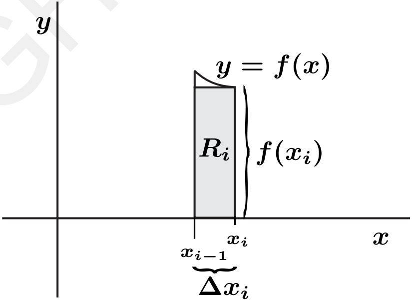
$$V = \sum_{i=1}^{n} V_i.$$



Volumes by the Disk Method

Replace W_i by the rectangle R_i with height $f(x_i)$ and base on the interval $[x_{i-1}, x_i]$.

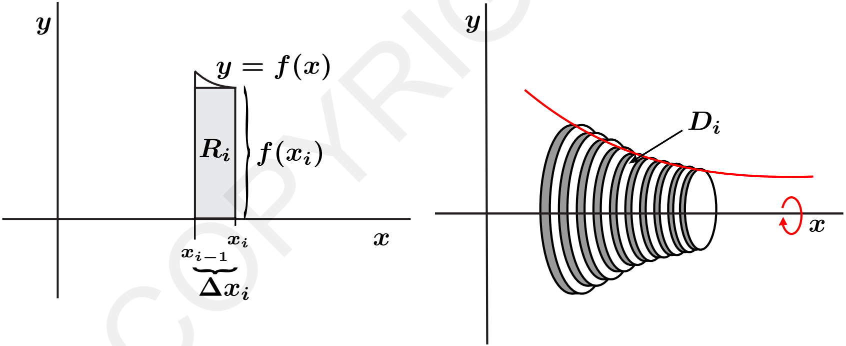




Volumes by the Disk Method

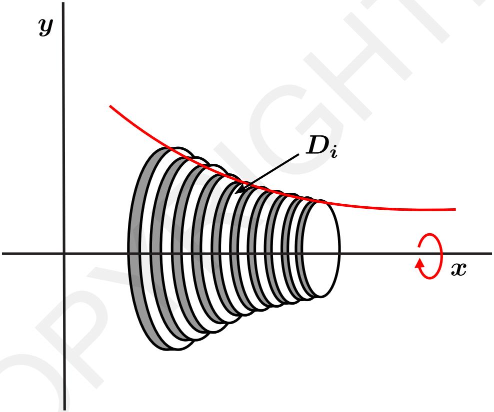
If $\triangle x_i$ is small, then V_i is approximately equal to the volume obtained by rotating R_i around the x-axis. Rotating each R_i generates a thin cylindrical disk D_i.

Therefore, the solid is approximated by a series of thin disks.



Volumes by the Disk Method

For this reason, this method to find the volume of revolution is often called the *Disk Method*.



The next step is to determine the volume V_i^* of the disk D_i.

Volumes by the Disk Method

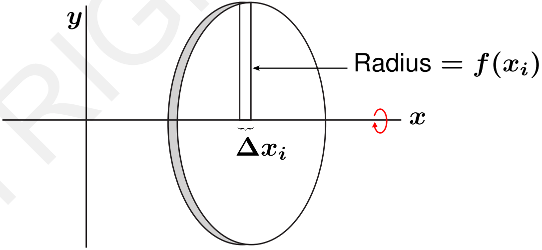
A close look at disk D_i shows that it has radius equal to the value of the function at x_i and its thickness is $\triangle x_i$.

Since the volume of a cylindrical disk is

$\pi \times (\text{radius})^2 \times (\text{thickness})$

we get that

$$V_i^* = \pi\, f(x_i)^2 \triangle x_i.$$



Volume of Disk $= \pi\, f(x_i)^2 \Delta x_i$

Volumes by the Disk Method

Then the approximation for the total volume of the solid of revolution is:

$$\begin{aligned} V &= \sum_{i=1}^{n} V_i \\ &\cong \sum_{i=1}^{n} V_i^* \\ &= \sum_{i=1}^{n} \pi\, f(x_i)^2 \triangle x_i \end{aligned}$$

It follows that

$$V \cong \sum_{i=1}^{n} \pi\, f(x_i)^2 \triangle x_i$$

and this is a Riemann sum for function $\pi\, f(x)^2$ over the interval $[a, b]$.

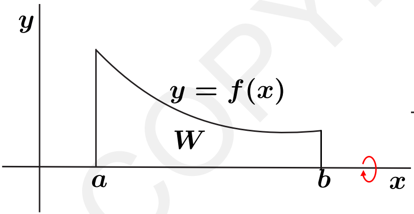
Letting $n \to \infty$, we achieve the formula for the volume of revolution.

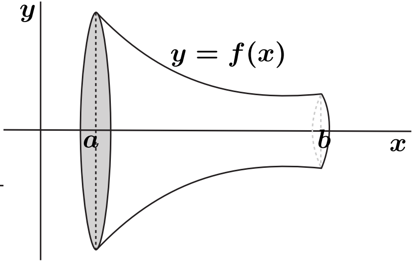
Volumes by the Disk Method

Volumes of Revolution: The Disk Method I

Let f be continuous on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. Let W be the region bounded by the graphs of f, the x-axis and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by

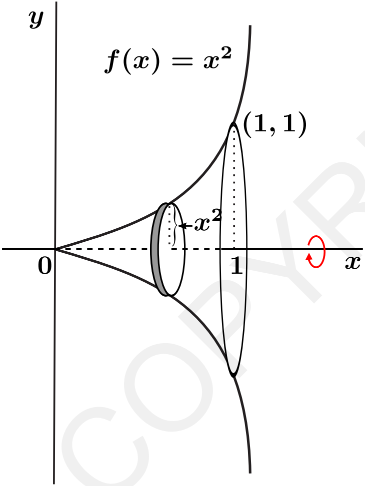
$$V = \int_a^b \pi f(x)^2 dx.$$





Volumes by the Disk Method

Example: Find the volume of the solid of revolution obtained by rotating the region bounded by the graph of the function $f(x) = x^2$, the x-axis, and the lines $x = 0$ and $x = 1$, around the x-axis.

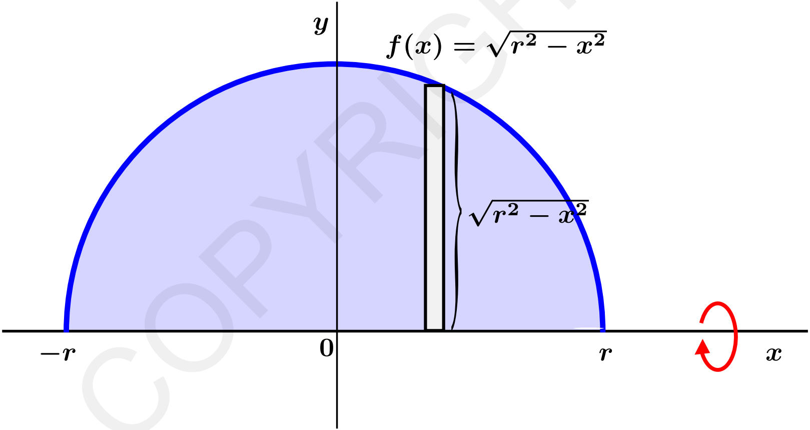


$$
\begin{aligned}
V &= \int_0^1 \pi f(x)^2\,dx \\
&= \int_0^1 \pi (x^2)^2\,dx \\
&= \pi \int_0^1 x^4\,dx \\
&= \pi \frac{x^5}{5}\Big|_0^1 \\
&= \frac{\pi}{5}
\end{aligned}
$$

Volumes by the Disk Method

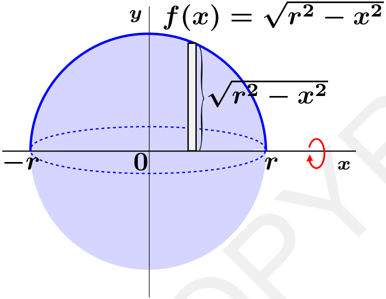
Example:

Find the volume of the sphere of radius r obtained by rotating the semi-circular region bounded by the graph of $f(x) = \sqrt{r^2 - x^2}$, the lines $x = -r, x = r$ and $y = 0$ around the x-axis.



Volumes by the Disk Method

Example (continued): Find the volume of the sphere of radius r obtained by rotating the semi-circular region bounded by the graph of $f(x) = \sqrt{r^2 - x^2}$, the lines $x = -r$, $x = r$ and $y = 0$ around the x-axis.



$$\begin{aligned} V &= \int_{-r}^{r} \pi f(x)^2 dx \\ &= \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx \\ &= \int_{-r}^{r} \pi (r^2 - x^2)\, dx \\ &= \pi \left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} \\ &= \pi \left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 - \frac{(-r)^3}{3}\right)\right) \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

which is the general formula for the volume of a sphere.