Areas Between Curves

Created by

Barbara Forrest and Brian Forrest

Areas Under Curves

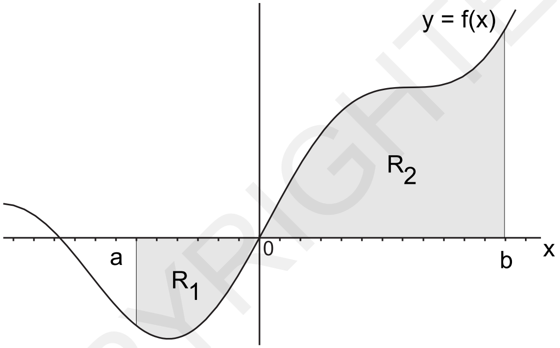
Recall:

In general, if f is a continuous function on the interval $[a, b]$, then

$$\int_a^b f(x)\ dx$$

represents the area of the region under the graph of f that lies *above* the x-axis between $x = a$ and $x = b$ ***minus*** the area of the region above the graph of f that lies *below* the x-axis between $x = a$ and $x = b$.

Areas Under Curves



$$\int_a^b f(x)\,dx = \text{Area } R_2 - \text{Area } R_1$$

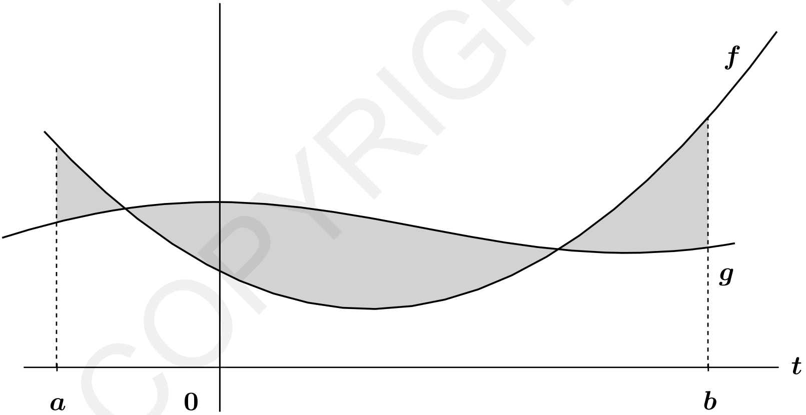
Areas Between Curves

Question: How do you calculate the area located **between** two curves?

Areas Between Curves

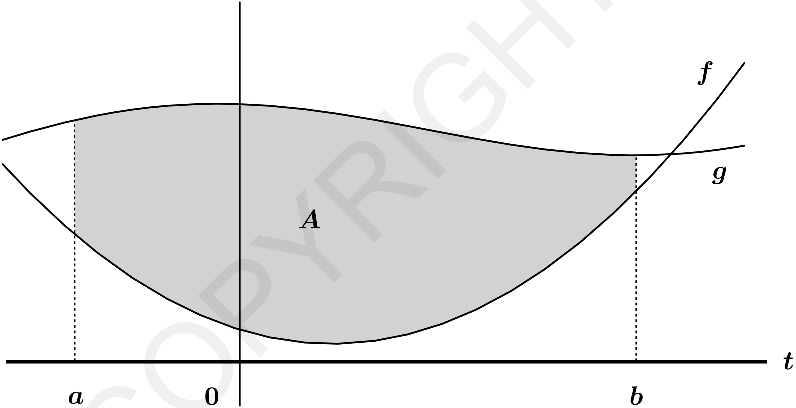
Problem:

Let f and g be continuous on an interval $[a, b]$. Find the area of the region bounded by the graphs of the two functions, f and g, and the lines $t = a$ and $t = b$.



Areas Between Curves

Let's begin with a simple case. Assume that $f(t) \le g(t)$ for all $t \in [a, b]$. The task is to find the area A of the region in the diagram.

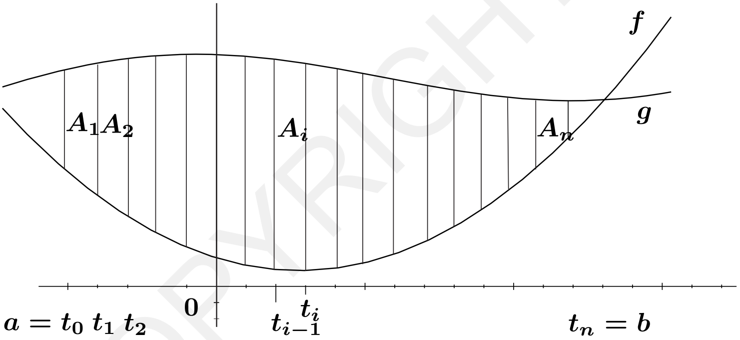


Areas Between Curves

Construct a regular n-partition

$$a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b$$

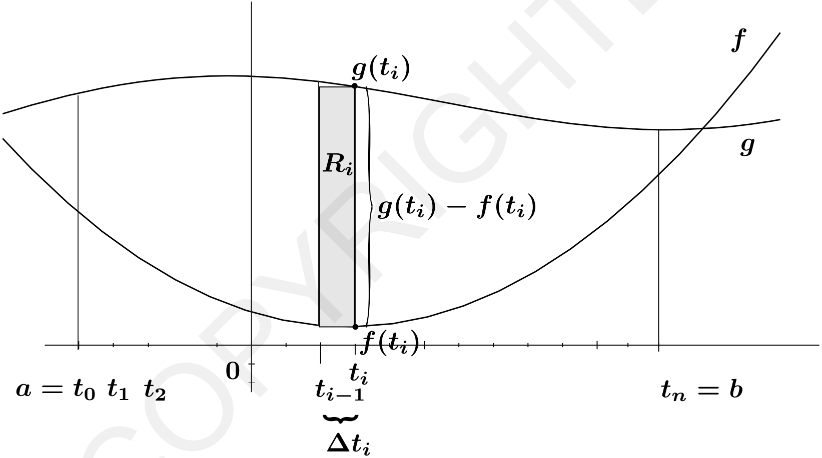
with $\Delta t_i = \frac{b-a}{n}$ and $t_i = a + \frac{i(b-a)}{n}$.



This partition divides A into n subregions which we label as $A_1, A_2, \cdots, A_n$ where A_i is the region bounded by the graphs f and g, and the lines $t = t_{i-1}$ and $t = t_i$.

Areas Between Curves

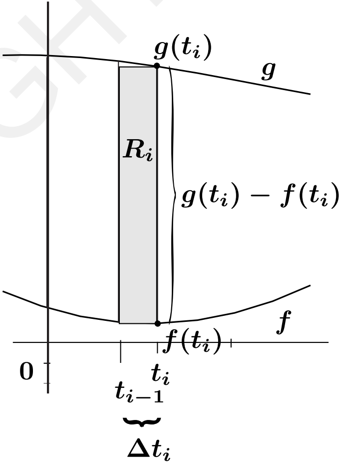
A rectangle R_i can now be used to estimate the area A_i.



Areas Between Curves

The height of the rectangle R_i is $h = g(t_i) - f(t_i)$ and its width is $\Delta t_i = \frac{b-a}{n}$, so the area A_i is estimated by

$$A_i \cong (g(t_i) - f(t_i))\, \Delta t_i.$$



Areas Between Curves

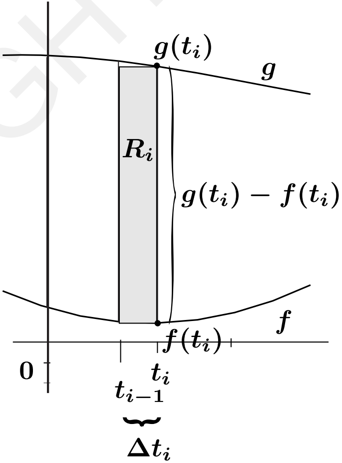
The height of the rectangle R_i is $h = g(t_i) - f(t_i)$ and its width is $\Delta t_i = \frac{b-a}{n}$, so the area A_i is estimated by

$$A_i \cong (g(t_i) - f(t_i))\,\Delta t_i.$$

Thus

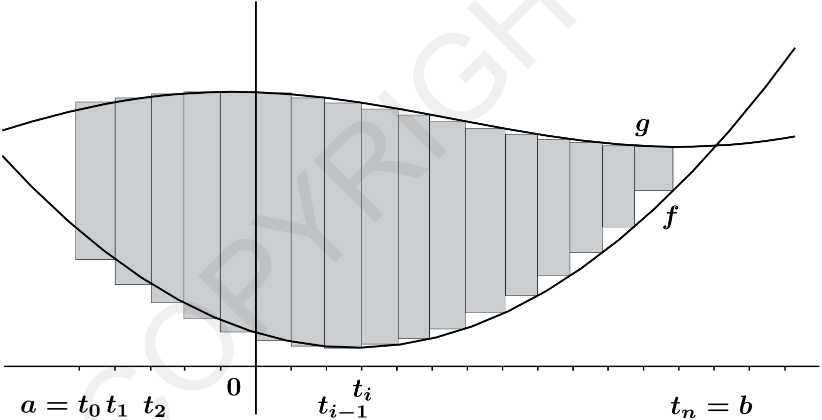
$$\begin{aligned} A &= \sum_{i=1}^{n} A_i \\ &\cong \sum_{i=1}^{n} (g(t_i) - f(t_i))\Delta t_i \\ &\cong \sum_{i=1}^{n} (g(t_i) - f(t_i))\frac{b-a}{n} \end{aligned}$$

with the latter sum equal to a right-hand Riemann sum for the function $g - f$ on $[a, b]$.



Areas Between Curves

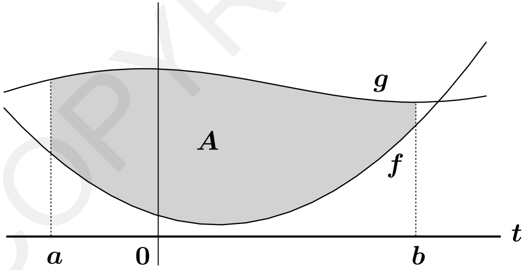
$$A \cong \sum_{i=1}^{n} (g(t_i) - f(t_i)) \frac{b-a}{n}$$



Areas Between Curves

Let $n \to \infty$. Then

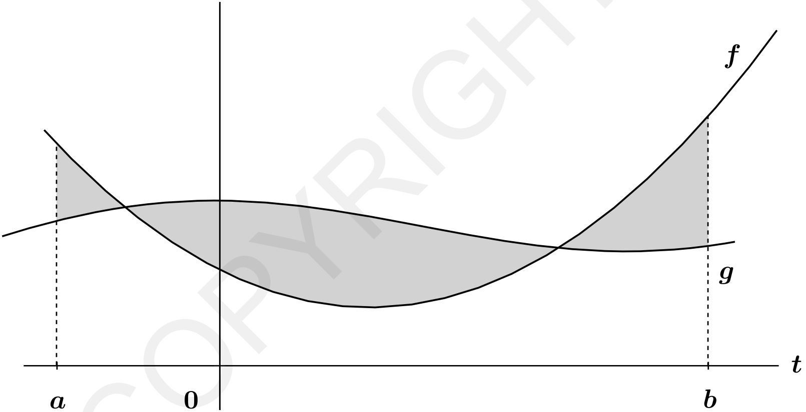
$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} (g(t_i) - f(t_i))\Delta t_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (g(t_i) - f(t_i))\frac{b-a}{n} \\ &= \int_a^b (g(t) - f(t))\, dt \end{aligned}$$



Areas Between Curves : General Case

Question:

What if f and g cross at one or more locations on the interval $[a, b]$?

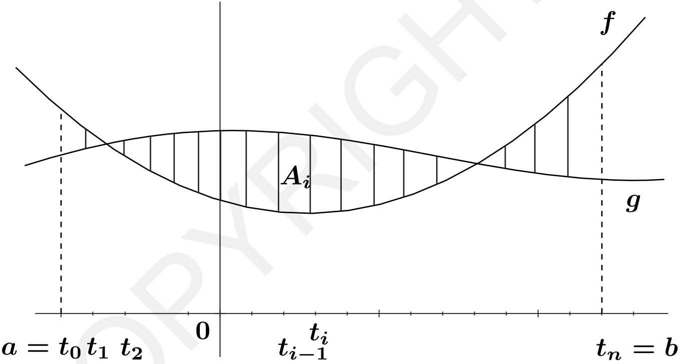


Areas Between Curves : General Case

Again construct a regular n-partition

$$a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b$$

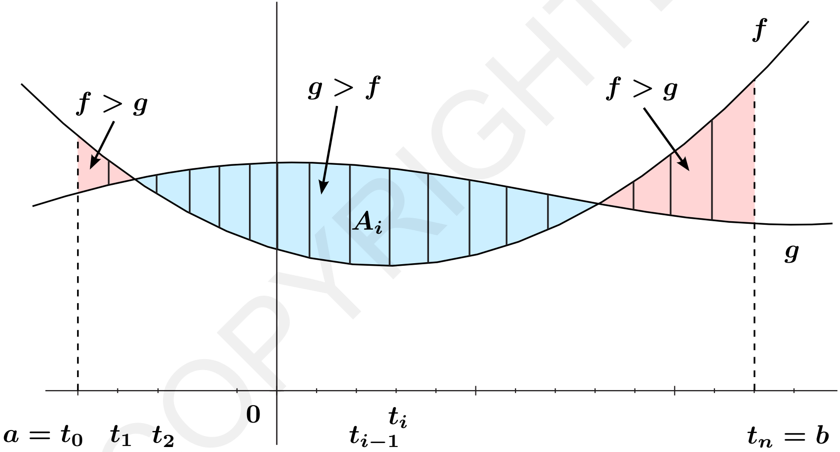
with $\Delta t_i = \frac{b-a}{n}$ and $t_i = a + \frac{i(b-a)}{n}$.



This divides A into n subregions $A_1, A_2, \cdots, A_n$ where A_i is the region bounded by the graphs f and g, and the lines $t = t_{i-1}$ and $t = t_i$.

Areas Between Curves : General Case

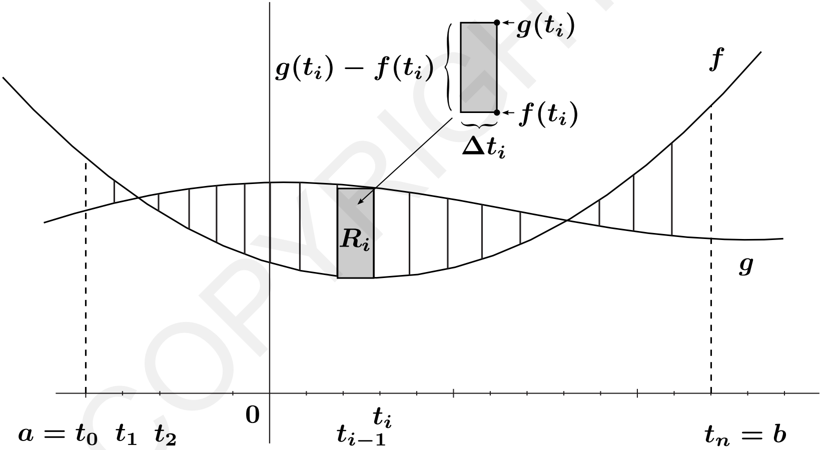
We must be concerned with whether $f(t_i) \le g(t_i)$ or $g(t_i) \le f(t_i)$.



Areas Between Curves : General Case

Case 1: If $f(t_i) \leq g(t_i)$, then the height of the rectangle R_i is $h_i = g(t_i) - f(t_i)$ and its width is $\Delta t_i = \frac{b-a}{n}$. That is

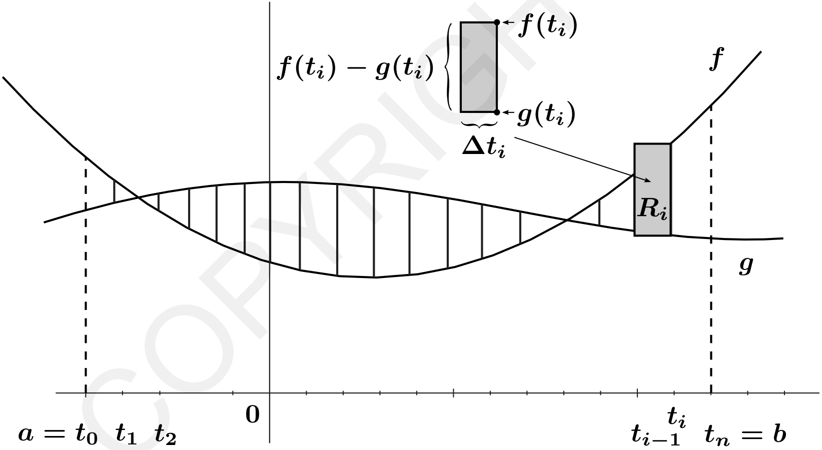
$$A_i \cong (g(t_i) - f(t_i))\, \Delta t_i$$



Areas Between Curves : General Case

Case 2: If $g(t_i) \leq f(t_i)$, then the height of the rectangle is now $h_i = f(t_i) - g(t_i)$. The width remains as $\Delta t_i = \frac{b-a}{n}$, so

$$A_i \cong (f(t_i) - g(t_i))\,\Delta t_i$$



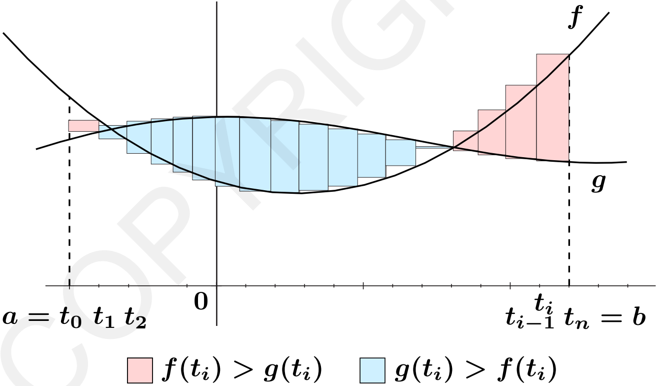
Areas Between Curves : General Case

To summarize, we have that

$$A_i \cong h_i \, \Delta t_i$$

where

$$h_i = \begin{cases} g(t_i) - f(t_i) & \text{if} \quad g(t_i) - f(t_i) \geq 0 \\ f(t_i) - g(t_i) & \text{if} \quad g(t_i) - f(t_i) < 0 \end{cases}$$



Areas Between Curves : General Case

The estimate for the area between f and g on $[a, b]$ is

$$\begin{aligned} A &= \sum_{i=1}^{n} A_i \\ &\cong \sum_{i=1}^{n} h_i \, \Delta t_i \end{aligned}$$

However, since

$$h_i = \begin{cases} g(t_i) - f(t_i) & \text{if} \quad g(t_i) - f(t_i) \geq 0 \\ f(t_i) - g(t_i) & \text{if} \quad g(t_i) - f(t_i) < 0 \end{cases}$$

then h_i is equivalent to

$$h_i = \mid g(t_i) - f(t_i) \mid .$$

Areas Between Curves : General Case

The total area is

$$\begin{aligned} A &= \sum_{i=1}^{n} A_i \\ &\cong \sum_{i=1}^{n} \mid g(t_i) - f(t_i) \mid \Delta t_i \end{aligned}$$

This is a right-hand Riemann sum for the function $\mid g - f \mid$ on $[a, b]$.

Let $n \to \infty$, then

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \mid g(t_i) - f(t_i) \mid \Delta t_i \\ &= \int_a^b \mid g(t) - f(t) \mid dt \end{aligned}$$

Areas Between Curves : General Case

Area Between Curves

Let f and g be continuous on $[a, b]$. Let A be the region bounded by the graphs of $f$and g, the line $t = a$ and the line $t = b$. Then the area of region A is given by

$$A = \int_a^b \mid g(t) - f(t) \mid \, dt.$$