Areas Between Curves: Examples

Created by

Barbara Forrest and Brian Forrest

Areas Under Curves: Examples

Recall:

Area Between Curves

Let f and g be continuous on $[a, b]$. Let A be the region bounded by the graphs of f and g, the line $t = a$ and the line $t = b$. Then the area of region A is given by

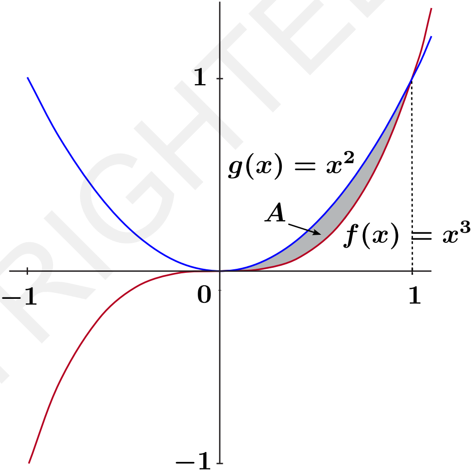
$$A = \int_a^b \mid g(t) - f(t) \mid \, dt.$$

Areas Under Curves: Examples

Example:

Find the area A of the closed region bounded by the graphs of the functions $g(x) = x^2$ and $f(x) = x^3$.

This area is the shaded region in the diagram.



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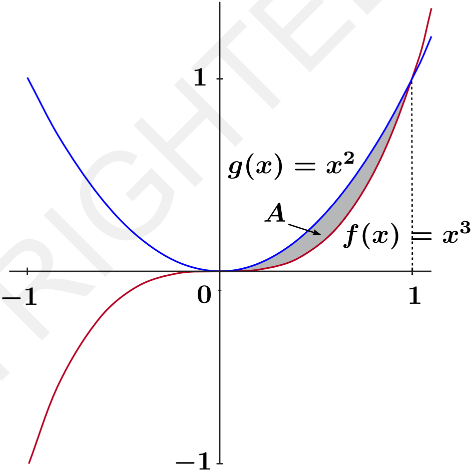
Example (continued):

The graphs cross when $x^3 = x^2$ or when

$$\begin{aligned} 0 &= x^3 - x^2 \\ \Rightarrow \qquad 0 &= x^2(x-1) \end{aligned}$$

This occurs when $x = 0$ and $x = 1$.

The area is bounded by the functions $g(x) = x^2$ and $f(x) = x^3$ between the lines $x = 0$ and $x = 1$.



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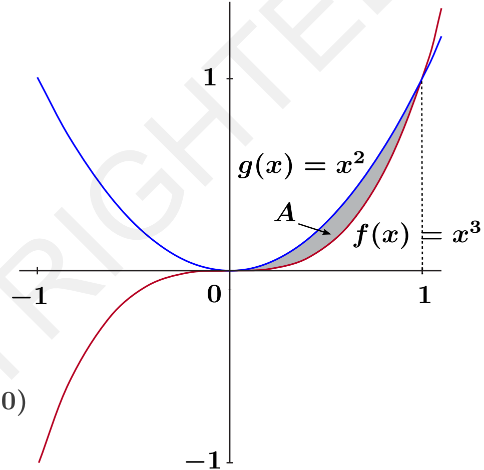
Example (continued):

Notice that $x^2 \geq x^3$ on the interval $[0, 1]$.

Then the area is

$$\begin{aligned} A &= \int_0^1 \left(x^2 - x^3\right)\, dx \\ &= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 \\ &= \left(\frac{1}{3} - \frac{1}{4}\right) - (0 - 0) \\ &= \frac{1}{12} \end{aligned}$$



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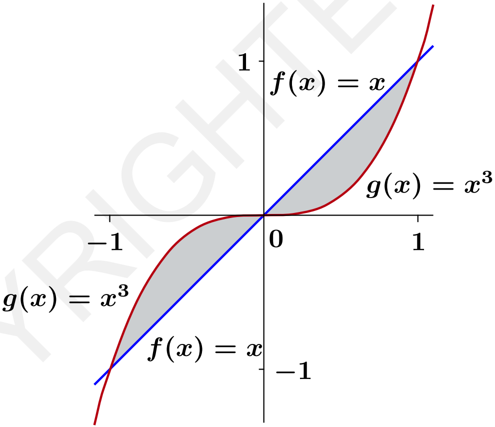
Example:

Find the total area A of the closed regions bounded by the graphs of the functions $f(x) = x$ and $g(x) = x^3$.

The shaded regions in the diagram represent A.



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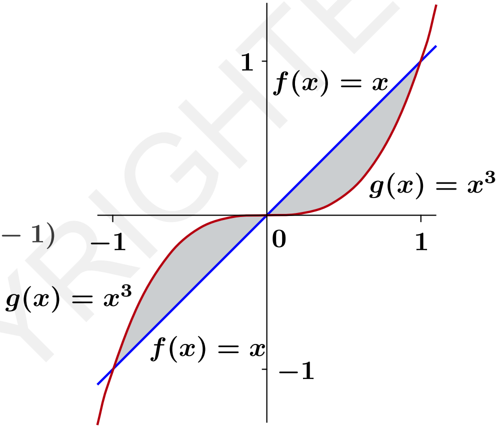
Example (continued):

Points of Intersection

The graphs intersect where $x^3 = x$.

$$\begin{aligned} 0 &= x^3 - x \\ \Rightarrow 0 &= x(x^2 - 1) \\ \Rightarrow 0 &= x(x+1)(x-1) \end{aligned}$$



The points of intersection occur at $x = -1$, $x = 0$, and $x = 1$.

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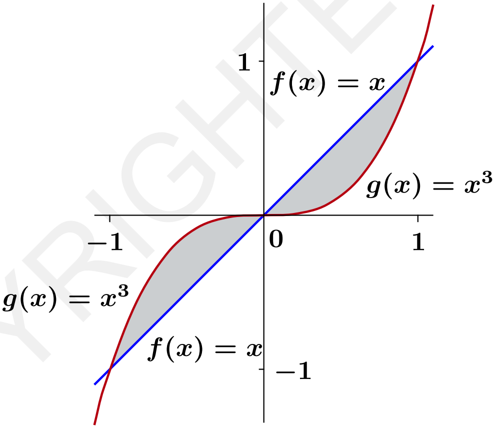
Example (continued):

We **cannot** apply the Fundamental Theorem of Calculus directly to $\mid x^3 - x \mid$ to calculate the area since f and g intersect on the interval $[-1, 1]$.



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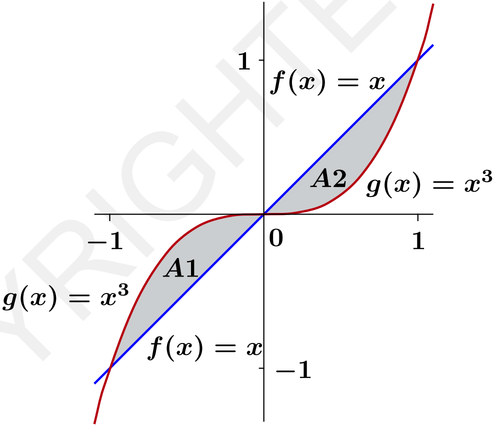
Example (continued):

Instead, we must consider the area in two parts, $A1$ and $A2$.



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Example (continued):

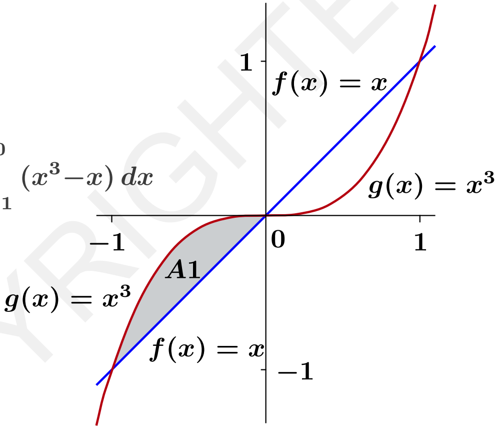
Case: Area of A1

On the interval $[-1, 0]$

$$x^3 \geq x$$

so

$$\int_{-1}^{0} \mid x^3 - x \mid \, dx = \int_{-1}^{0} (x^3 - x) \, dx$$



This integral represents $A1$, the shaded area in the diagram.

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Example (continued):

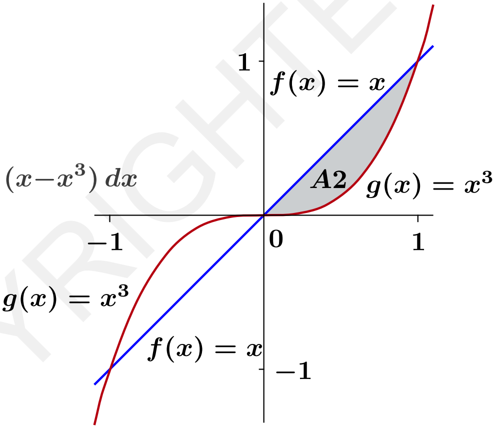
Case: Area of A2

On the interval $[0, 1]$

$$x \geq x^3$$

so

$$\int_0^1 \mid x^3 - x \mid \, dx = \int_0^1 (x - x^3) \, dx$$



This integral represents $A2$, the shaded area in the diagram.

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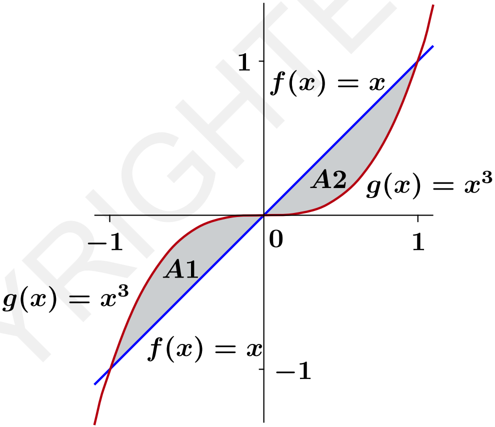
Example (continued):

Total Area
Between the Curves

The total area A between the curves $f(x) = x$ and $g(x) = x^3$ on the interval $[-1, 1]$ is

$$A = A1 + A2.$$



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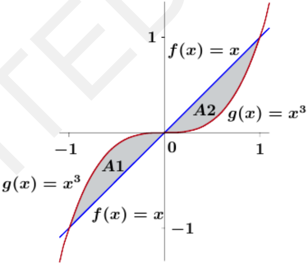
Example (continued):

Total Area Between the Curves



$$
\begin{aligned}
A &= \int_{-1}^{1} |\, x^3 - x \,|\; dx \\
&= A1 + A2 \\
&= \int_{-1}^{0} |\, x^3 - x \,|\; dx + \int_{0}^{1} |\, x^3 - x \,|\; dx \\
&= \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} (x - x^3)\, dx \\
&= \left(\frac{x^4}{4} - \frac{x^2}{2} \right) \Bigg|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Bigg|_{0}^{1} \\
&= \left((0 - 0) - \left(\frac{1}{4} - \frac{1}{2} \right) \right) + \left(\left(\frac{1}{2} - \frac{1}{4} \right) - (0 - 0) \right) \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}
$$