Arc Length

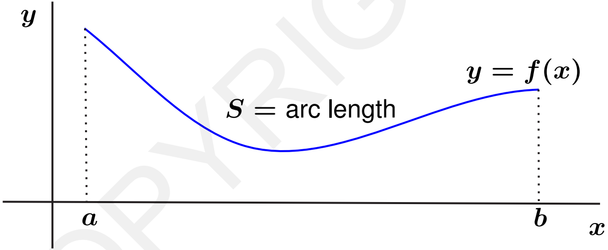
Created by

Barbara Forrest and Brian Forrest

Arc Length

Problem:

Let f be continuously differentiable on $[a, b]$. What is the arc length S of the graph of f on the interval $[a, b]$?

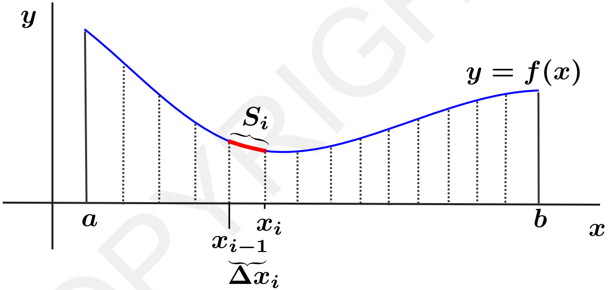


Arc Length

Let

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

be a regular n-partition of $[a, b]$.
Let S_i denote the length of the arc joining $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$.

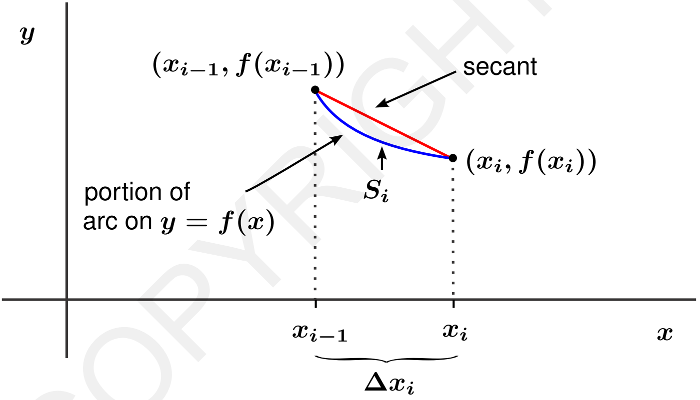


Then the length of the graph of f on the interval $[a, b]$ is

$$S = \sum_{i=1}^{n} S_i.$$

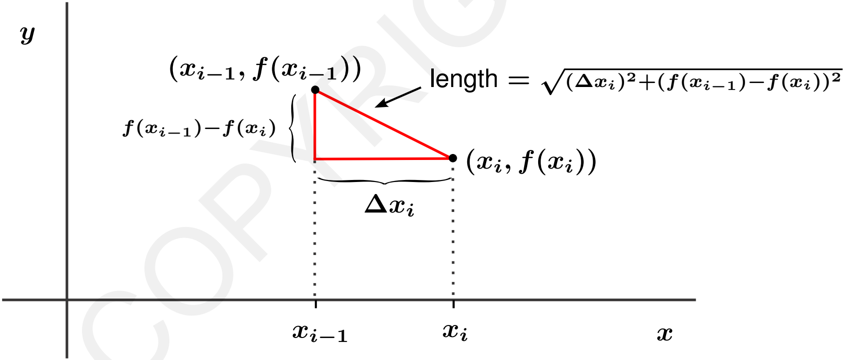
Arc Length

Observe that if $\triangle x_i$ is small, then S_i is approximately equal to the length of the secant line joining $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$.



Arc Length

$$
\begin{aligned}
S_i &\cong \sqrt{(\triangle x_i)^2 + (\triangle y_i)^2} \\
&= \sqrt{(\triangle x_i)^2 + (f(x_i) - f(x_{i-1}))^2}
\end{aligned}
$$



Arc Length

Applying the Mean Value Theorem guarantees a $c_i \in (x_{i-1}, x_i)$ so

$$f(x_i) - f(x_{i-1}) = f'(c_i) \triangle x_i.$$

Therefore,

$$\begin{aligned} S_i &\cong \sqrt{(\triangle x_i)^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sqrt{(\triangle x_i)^2 + (f'(c_i)\triangle x_i)^2} \\ &= \sqrt{(\triangle x_i)^2 + (f'(c_i))^2(\triangle x_i)^2} \\ &= \sqrt{(\triangle x_i)^2(1 + (f'(c_i))^2)} \\ &= \sqrt{1 + (f'(c_i))^2}\,\triangle x_i \end{aligned}$$

Hence,

$$\begin{aligned} S &= \sum_{i=1}^{n} S_i \\ &\cong \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2}\,\triangle x_i \end{aligned}$$

Arc Length

Note that

$$S \cong \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2}\, \triangle x_i$$

is a Riemann sum for the function $\sqrt{1 + (f\,'(x))^2}$ over the interval $[a, b]$.

Therefore, letting $n \to \infty$, we get

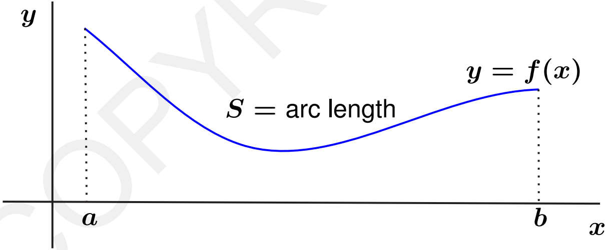
$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Arc Length

Arc Length

Let f be continuously differentiable on $[a, b]$. Then the arc length S of the graph of f over the interval $[a, b]$ is given by

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$



Arc Length

Example:

Find the length S of the portion of the graph of the function $f(x) = \frac{2x^{\frac{3}{2}}}{3}$ between $x = 1$ and $x = 2$.

In this case, $f'(x) = x^{\frac{1}{2}}$.

$$
\begin{aligned}
S &= \int_1^2 \sqrt{1 + (f'(x))^2}\, dx \\
&= \int_1^2 \sqrt{1 + (x^{\frac{1}{2}})^2}\, dx \\
&= \int_1^2 \sqrt{1 + x}\, dx \\
&= \left. \frac{2(1+x)^{\frac{3}{2}}}{3} \right|_1^2 \\
&= \frac{2(3)^{\frac{3}{2}}}{3} - \frac{2(2)^{\frac{3}{2}}}{3} \\
&= \frac{2}{3}(3^{\frac{3}{2}} - 2^{\frac{3}{2}}) \\
&\cong 1.578
\end{aligned}
$$