Type II Improper Integrals

Created by

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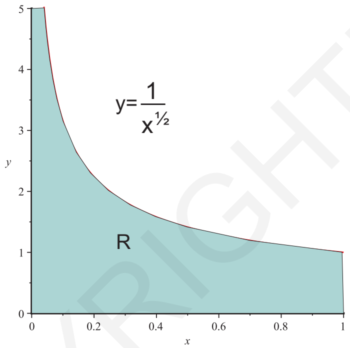
Type II Improper Integrals

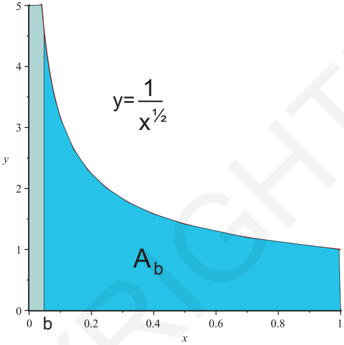


Question: What is the area of the unbounded region R under the graph of

$$f(x) = \frac{1}{x^{\frac{1}{2}}}$$

above the x-axis and between the lines $x = 0$ and $x = 1$? Is the area infinite?

Type II Improper Integrals



Strategy: Pick a small $b > 0$ and consider

$$\begin{aligned} A_b &= \int_b^1 \frac{1}{x^{\frac{1}{2}}}\,dx \\ &= 2x^{\frac{1}{2}}\Big|_b^1 \\ &= 2(1-\sqrt{b}) \end{aligned}$$

Note:

$$\lim_{b\to 0+} A_b = \lim_{b\to 0+} 2(1-\sqrt{b}) = 2$$

Type II Improper Integrals

Definition: [Type II Improper Integrals]

1) Let f be integrable on $[t, b]$ for every $t \in (a, b]$ with either $\lim_{x \to a^+} f(x) = \infty$ or $\lim_{x \to a^+} f(x) = -\infty$. We say that the *Type II improper integral*

$$\int_a^b f(x)\, dx$$

converges if

$$\lim_{t \to a^+} \int_t^b f(x)\, dx$$

exists.
In this case, we write

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx$$

Otherwise, we say that $\int_a^b f(x)\, dx$ *diverges*.

Type II Improper Integrals

Definition: [Type II Improper Integrals]

2) Let f be integrable on on $[a, t]$ for every $t \in [a, b)$ with either $\lim_{x \to b^-} f(x) = \infty$ or $\lim_{x \to b^-} f(x) = -\infty$. We say that the *Type II improper integral*

$$\int_a^b f(x)\, dx$$

converges if

$$\lim_{t \to b^-} \int_a^t f(x)\, dx$$

exists.
In this case, we write

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx$$

Otherwise, we say that $\int_a^b f(x)\, dx$ *diverges*.

p-Test for Type II Improper Integrals

Remark: In determining the convergence or divergence of a Type I improper integral, the p-test was an important tool. There is a natural analog of the p-test for Type II improper integrals.

Theorem: [p-Test for Type II Improper Integrals]

The improper integral

$$\int_0^1 \frac{1}{x^p}\,dx$$

converges if and only if $p < 1$.

If $p < 1$, then

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}$$

p-Test for Type II Improper Integrals

Proof of the p-Test:

First assume that $p \neq 1$. By definition

$$\begin{aligned} \int_0^1 \frac{1}{x^p}\,dx &= \lim_{t\to 0^+} \int_t^1 \frac{1}{x^p}\,dx \\ &= \lim_{t\to 0^+} \frac{1}{1-p} x^{1-p}\Big|_t^1 \\ &= \lim_{t\to 0^+} \frac{1}{1-p} - \frac{1}{1-p} t^{1-p} \end{aligned}$$

Now if $p < 1$, then $1 - p > 0$ so

$$\lim_{t\to 0^+} \frac{1}{1-p} - \frac{1}{1-p} t^{1-p} = \frac{1}{1-p}$$

But if $p > 1$, then $1 - p < 0$ so

$$\lim_{t\to 0^+} \frac{1}{1-p} - \frac{1}{1-p} t^{1-p} = \infty$$

p-Test for Type II Improper Integrals

Proof of the p-Test (continued):

Finally, if $p = 1$, then

$$\begin{aligned} \int_0^1 \frac{1}{x}\,dx &= \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx \\ &= \lim_{t\to 0^+} \ln(x)\Big|_t^1 \\ &= \lim_{t\to 0^+} \ln(1) - \ln(t) \\ &= \infty \end{aligned}$$