

Partial Fractions

(Part 3)

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Type III Partial Fraction Decomposition

Remark: Not all polynomials factor over the Real numbers into products of linear terms. For example, the polynomial $x^2 + 1$ cannot be factored any further.

This is an example of an *irreducible quadratic*. In fact, a quadratic $ax^2 + bx + c$ is irreducible if its discriminant

$$b^2 - 4ac < 0$$

Fact: The *Fundamental Theorem of Algebra* shows that every polynomial $q(x)$ factors in the form

$$q(x) = a(p_1(x))^{m_1}(p_2(x))^{m_2}(p_3(x))^{m_3} \cdots (p_k(x))^{m_k}$$

where each $p_i(x)$ is either of the form $(x - a)$ or it is an irreducible quadratic of the form $x^2 + bx + c$.

Type III Partial Fraction Decomposition

Definition: [Type III Partial Fraction Decomposition]

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function with

$\text{degree}(p(x)) < \text{degree}(q(x))$, but $q(x)$ does not factor into linear terms.

We say that f admits a *Type III partial fraction decomposition* as follows:

Suppose that $q(x)$ has an irreducible factor $x^2 + bx + c$ as above with multiplicity m . Then this factor will contribute terms of the form

$$\frac{B_1x + C_1}{x^2 + bx + c} + \frac{B_2x + C_2}{(x^2 + bx + c)^2} + \cdots + \frac{B_mx + C_m}{(x^2 + bx + c)^m}$$

to the decomposition.

The linear terms are handled exactly as they were in the previous cases.

Type III Partial Fraction Decomposition

Example: Evaluate $\int \frac{1}{x^3+x} dx$.

Solution: First observe that

$$f(x) = \frac{1}{x^3+x} = \frac{1}{x(x^2+1)}$$

so there will be constants A , B and C such that

$$f(x) = \frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Type III Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^3+x} dx$.

Step 1: To find the constants, we begin by cross multiplying to obtain

$$1 = A(x^2 + 1) + (Bx + C)x \quad (*)$$

Step 2: Substitute $x = 0$, the only Real root, to find the coefficient A .

This gives

$$1 = A(0^2 + 1) + (B(0) + C)(0)$$

or

$$A = 1$$

Type III Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^3+x} dx$.

Step 3: The remaining constants are found by comparing coefficients.

Expanding equation (*) gives:

$$1 = (A + B)x^2 + Cx + A$$

Comparing the coefficients of x^2 gives

$$0 = A + B \Rightarrow B = -A$$

Since $A = 1$, we get

$$B = -1$$

Comparing the coefficients of x gives

$$C = 0$$

Therefore,

$$\frac{1}{x(x^2 + 1)} = \frac{1}{x} - \frac{x}{x^2 + 1}$$

Type III Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^3+x} dx$.

Step 4: We have

$$\begin{aligned}\int \frac{1}{x^3+x} dx &= \int \frac{1}{x(x^2+1)} dx \\ &= \int \frac{1}{x} dx - \int \frac{x}{x^2+1} dx \\ &= \ln(|x|) - \int \frac{x}{x^2+1} dx\end{aligned}$$

Type III Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^3+x} dx$.

To finish the calculation, we use the substitution $u = x^2 + 1$, so $du = 2x dx$ and $dx = \frac{du}{2x}$ to get

$$\begin{aligned}\int \frac{x}{x^2 + 1} dx &= \int \frac{x}{u} \frac{du}{2x} \\ &= \frac{1}{2} \int \frac{1}{u} du \\ &= \frac{1}{2} \ln(|u|) + c \\ &= \frac{1}{2} \ln(x^2 + 1) + c\end{aligned}$$

Putting this all together gives

$$\int \frac{1}{x^3 + x} dx = \ln(|x|) - \frac{1}{2} \ln(x^2 + 1) + c$$

Type III Partial Fraction Decomposition

Remarks:

- 1) Because $C = 0$ in the previous example,

$$f(x) = \frac{1}{x^3 + x} = \frac{1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

and the integral was straightforward. Unfortunately, this is not the case for most Type III partial fractions.

- 2) All of the partial fraction decompositions required the rational function to satisfy the condition $\text{degree}(p(x)) < \text{degree}(q(x))$. If this is not the case, then we need to use long division of polynomials to find polynomials $r(x)$ and $p_1(x)$ such that

$$\frac{p(x)}{q(x)} = r(x) + \frac{p_1(x)}{q(x)}$$

and $\text{degree}(p_1(x)) < \text{degree}(q(x))$.