Partial Fractions
(Part 2)

Created by

Barbara Forrest and Brian Forrest

Type II Partial Fraction Decomposition

Definition: [Type II Partial Fraction Decomposition]

Assume that

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials such that

1. $degree(p(x)) < degree(q(x)) = k$,
2. $q(x)$ can be factored into the product of linear terms with non-distinct roots

   $$q(x) = a(x - a_1)^{m_1}(x - a_2)^{m_2}(x - a_3)^{m_3} \cdots (x - a_l)^{m_l}$$

   where at least one of the m_j's is greater than 1.

We say that f admits a *Type II partial fraction decomposition* as follows:

Each expression $(x - a_j)^{m_j}$ in the factorization of $q(x)$ will contribute m_j terms to the decomposition, one for each power of $x - a_j$ from 1 to m_j, which when combined will be of the form

$$\frac{p(x)}{q(x)} = \sum_{j=1}^{l} \frac{A_{j,1}}{x - a_j} + \frac{A_{j,2}}{(x - a_j)^2} + \frac{A_{j,3}}{(x - a_j)^3} + \cdots + \frac{A_{j,m_j}}{(x - a_j)^{m_j}}$$

The number m_j is called the *multiplicity* of the root a_j.

Type II Partial Fraction Decomposition

Observation: The procedure for integrating a rational function with a Type II partial fraction decomposition is very similar to the procedure for integrating those of Type I.

Example: Evaluate $\int \frac{1}{x^2(x+1)}\,dx$.

We have

$$\frac{1}{x^2(x+1)} = \frac{A_{1,1}}{x} + \frac{A_{1,2}}{x^2} + \frac{A_{2,1}}{x+1}$$

$$\begin{aligned} \int \frac{1}{x^2(x+1)}\,dx &= \int \frac{A_{1,1}}{x}\,dx + \int \frac{A_{1,2}}{x^2}\,dx + \int \frac{A_{2,1}}{x+1}\,dx \\ &= A_{1,1}\ln(|\,x\,|) - \frac{A_{1,2}}{x} + A_{2,1}\ln(|\,x+1\,|) + C \end{aligned}$$

Problem: How do we find $A_{1,1}$, $A_{1,2}$ and $A_{2,1}$?

Type II Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^2(x+1)}\,dx$.

If

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1},$$

then multiplying by $q(x)$ gives:

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

Letting $x = 0$ gives

$$1 = B(0+1) \Rightarrow B = 1$$

Letting $x = -1$ gives

$$1 = C(-1)^2 \Rightarrow C = 1$$

Question: How do we find A?

Type II Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^2(x+1)}\,dx$.

Observation: We find A by comparing coefficients in the equation

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

We have

$$1 = Ax^2 + Ax + Bx + B + Cx^2 = (A+1)x^2 + (A+1)x + 1$$

$$\begin{aligned} &\Rightarrow \quad 0x^2 = (A+1)x^2 \\ &\Rightarrow \quad 0 = A+1 \\ &\Rightarrow \quad A = -1 \end{aligned}$$

Hence

$$\int \frac{1}{x^2(x+1)}\,dx = -\ln(|\,x\,|) - \frac{1}{x} + \ln(|\,x+1\,|) + C$$

Type II Partial Fraction Decomposition

Remark: Not all polynomials factor over the Real numbers into products of linear terms. For example, the polynomial $x^2 + 1$ cannot be factored any further.

This is an example of an *irreducible quadratic*. In fact, a quadratic $ax^2 + bx + c$ is irreducible if its discriminant

$$b^2 - 4ac < 0$$

Fact: The *Fundamental Theorem of Algebra* shows that every polynomial $q(x)$ factors in the form

$$q(x) = a(p_1(x))^{m_1}(p_2(x))^{m_2}(p_3(x))^{m_3} \cdots (p_k(x))^{m_k}$$

where each $p_i(x)$ is either of the form $(x - a)$ or it is an irreducible quadratic of the form $x^2 + bx + c$.

Type III Partial Fraction Decomposition

Definition: [Type III Partial Fraction Decomposition]

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function with $degree(p(x)) < degree(q(x))$, but $q(x)$ does not factor into linear terms.

We say that f admits a *Type III partial fraction decomposition* as follows:

Suppose that $q(x)$ has an irreducible factor $x^2 + bx + c$ as above with multiplicity m. Then this factor will contribute terms of the form

$$\frac{B_1 x + C_1}{x^2 + bx + c} + \frac{B_2 x + C_2}{(x^2 + bx + c)^2} + \cdots + \frac{B_m x + C_m}{(x^2 + bx + c)^m}$$

to the decomposition.

The linear terms are handled exactly as they were in the previous cases.

Type III Partial Fraction Decomposition

Example: Evaluate $\int \frac{1}{x^3+x}\,dx$.

Solution: First observe that

$$f(x) = \frac{1}{x^3+x} = \frac{1}{x(x^2+1)}$$

so there will be constants A, B and C such that

$$f(x) = \frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Type III Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^3+x}\,dx$.

Step 1: To find the constants, we begin by cross multiplying to obtain

$$1 = A(x^2+1) + (Bx+C)x \qquad (*)$$

Step 2: Substitute $x = 0$, the only Real root, to find the coefficient A.

This gives

$$1 = A(0^2+1) + (B(0)+C)(0)$$

or

$$A = 1$$

Type III Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^3+x}\,dx$.

Step 3: The remaining constants are found by comparing coefficients.

Expanding equation $(*)$ gives:

$$1 = (A+B)x^2 + Cx + A$$

Comparing the coefficients of x^2 gives

$$0 = A + B \Rightarrow B = -A$$

Since $A = 1$, we get

$$B = -1$$

Comparing the coefficients of x gives

$$C = 0$$

Therefore,

$$\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$

Type III Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^3+x}\,dx$.

Step 4: We have

$$\begin{aligned}\int \frac{1}{x^3+x}\,dx &= \int \frac{1}{x(x^2+1)}\,dx \\ &= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx \\ &= \ln(|\,x\,|) - \int \frac{x}{x^2+1}\,dx\end{aligned}$$

Type III Partial Fraction Decomposition

Example (continued): Evaluate $\int \frac{1}{x^3+x}\,dx$.

To finish the calculation, we use the substitution $u = x^2 + 1$, so $du = 2x dx$ and $dx = \frac{du}{2x}$ to get

$$\begin{aligned} \int \frac{x}{x^2+1}\,dx &= \int \frac{x}{u}\frac{du}{2x} \\ &= \frac{1}{2}\int \frac{1}{u}\,du \\ &= \frac{1}{2}\ln(|\,u\,|) + c \\ &= \frac{1}{2}\ln(x^2+1) + c \end{aligned}$$

Putting this all together gives

$$\int \frac{1}{x^3+x}\,dx = \ln(|\,x\,|) - \frac{1}{2}\ln(x^2+1) + c$$

Type III Partial Fraction Decomposition

Remarks:

1) Because $C = 0$ in the previous example,

$$f(x) = \frac{1}{x^3 + x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

and the integral was straightforward. Unfortunately, this is not the case for most Type III partial fractions.

2) All of the partial fraction decompositions required the rational function to satisfy the condition $degree(p(x)) < degree(q(x))$. If this is not the case, then we need to use long division of polynomials to find polynomials $r(x)$ and $p_1(x)$ such that

$$\frac{p(x)}{q(x)} = r(x) + \frac{p_1(x)}{q(x)}$$

and $degree(p_1(x)) < degree(q(x))$.