Partial Fractions

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Partial Fractions

Recall: A function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials is called a *rational function*.

Problem: Evaluate

$$\int f(x)\,dx = \int \frac{p(x)}{q(x)}\,dx$$

Example

Example: Evaluate $\int \frac{1}{x^2-1}\,dx$.

Strategy: Look for constants A and B so that

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

then

$$\begin{aligned}
\int \frac{1}{x^2-1}\,dx &= \int \frac{1}{(x+1)(x-1)}\,dx \\
&= \int \left(\frac{A}{x+1} + \frac{B}{x-1}\right) dx \\
&= \int \frac{A}{x+1}\,dx + \int \frac{B}{x-1}\,dx \\
&= A\ln(|x+1|) + B\ln(|x-1|) + C
\end{aligned}$$

Example

Example (continued): Evaluate $\int \frac{1}{x^2-1}\,dx$..

Question: How do we find A and B?

Observation: We have

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)},$$

so

$$1 = A(x-1) + B(x+1)$$

Let

$$x = 1 \Rightarrow 1 = A(1-1) + B(1+1) \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$$

Let

$$x = -1 \Rightarrow 1 = A(-1-1) + B(-1+1) \Rightarrow 1 = -2A \Rightarrow A = -\frac{1}{2}$$

Example

Example (continued): Evaluate $\int \frac{1}{x^2-1}\,dx$.

Therefore

$$\frac{1}{(x+1)(x-1)} = \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}$$

and

$$\int \frac{1}{x^2-1}\,dx = -\frac{1}{2}\ln(|x+1|) + \frac{1}{2}\ln(|x-1|) + C$$

Type I Partial Fraction Decomposition

Definition: [Type I Partial Fraction Decomposition]

Assume that

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials such that

1. $degree(p(x)) < degree(q(x)) = k$,
2. $q(x)$ can be factored into the product of k linear terms each with distinct roots. That is

   $$q(x) = a(x - a_1)(x - a_2)(x - a_3) \cdots (x - a_k)$$

   where the a_i's are all different and none of the a_i's are roots of $p(x)$.

Then there exists constants A_1, A_2, A_3, $\cdots$, A_k such that

$$f(x) = \frac{1}{a}\left[\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \frac{A_3}{x - a_3} + \cdots + \frac{A_k}{x - a_k}\right]$$

We say that f admits a *Type I partial fraction decomposition*.

Type I Partial Fraction Decomposition

Theorem: [Integration of Type I Partial Fractions]

Assume that $f(x) = \frac{p(x)}{q(x)}$ admits a Type I partial fraction decomposition of the form

$$f(x) = \frac{1}{a}\left[\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_k}{x-a_k}\right]$$

Then

$$\begin{aligned}\int f(x)\,dx &= \frac{1}{a}\left[\int \frac{A_1}{x-a_1}\,dx + \int \frac{A_2}{x-a_2}\,dx + \cdots \right.\\ &\quad \left. + \int \frac{A_k}{x-a_k}\,dx\right]\\ &= \frac{1}{a}\left[A_1 \ln(|\ x-a_1\ |) + A_2 \ln(|\ x-a_2\ |) + \cdots \right.\\ &\quad \left. + A_k \ln(|\ x-a_k\ |)\right] + C\end{aligned}$$

Type I Partial Fraction Decomposition

Example: We know that

$$\frac{x}{(x+2)(x-1)(x-2)} = \frac{A_1}{x+2} + \frac{A_2}{x-1} + \frac{A_3}{x-2}$$

so

$$x = A_1(x-1)(x-2) + A_2(x+2)(x-2) + A_3(x+2)(x-1)$$

Substituting for the roots $x = -2$, $x = 1$ and $x = 2$ gives

$$A_1 = \frac{-2}{(-2-1)(-2-2)} = -\frac{1}{6}, \qquad (x = -2)$$

$$A_2 = \frac{1}{(1+2)(1-2)} = -\frac{1}{3}, \qquad (x = 1)$$

and

$$A_3 = \frac{2}{(2+2)(2-1)} = \frac{1}{2}, \qquad (x = 2)$$

Hence

$$\int \frac{x}{(x+2)(x-1)(x-2)}\,dx = -\frac{1}{6}\ln(|x+2|) - \frac{1}{3}\ln(|x-1|) + \frac{1}{2}\ln(|x-2|) + C$$

Type I Partial Fraction Decomposition

Key Observation:

If

$$\frac{p(x)}{q(x)} = \frac{1}{a}\left[\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \frac{A_3}{x-a_3} + \cdots + \frac{A_k}{x-a_k}\right]$$

then

$$A_m = \frac{a \cdot p(a_m)}{\prod\limits_{\substack{j=1 \\ j \neq m}}^{k} (a_m - a_j)}$$

where

$$\prod_{j=1}^{k} b_j = b_1 \cdot b_2 \cdot b_3 \cdots b_k$$