Monotone Convergence Theorem for Functions

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Monotone Convergence Theorem for Functions

Recall:

Theorem: [Monotone Convergence Theorem (MCT)]

If $\{a_n\}$ is non-decreasing and bounded above, then $\{a_n\}$ converges.

Question: Is there an analog of the MCT for monotonic functions?

Monotone Convergence Theorem for Functions

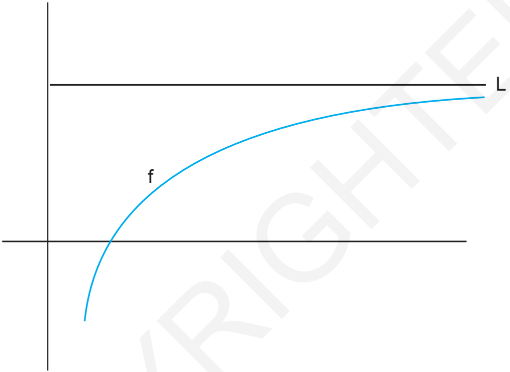
Theorem: [Monotone Convergence Theorem for Functions (MCTF)]

Assume that f is nondecreasing on $[a, \infty)$. Let

$$S = \{f(x) \mid x \in [a, \infty)\}$$

1) If S is bounded above, then $\lim_{x \to \infty} f(x) = L = lub(S)$.
2) If S is not bounded above, then $\lim_{x \to \infty} f(x) = \infty$.

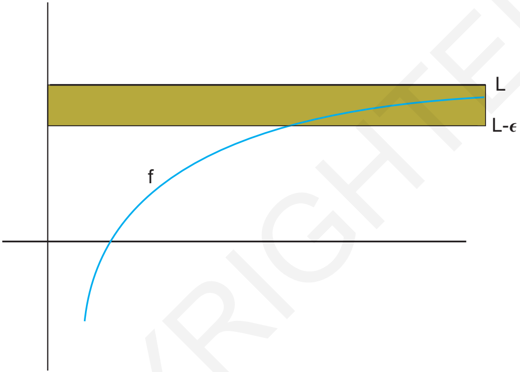
Monotone Convergence Theorem for Functions



Proof:

1) Assume that S is bounded. Let $L = lub(S)$.

Monotone Convergence Theorem for Functions



Proof:

1) Assume that S is bounded. Let $L = lub(S)$. Let $\epsilon > 0$.

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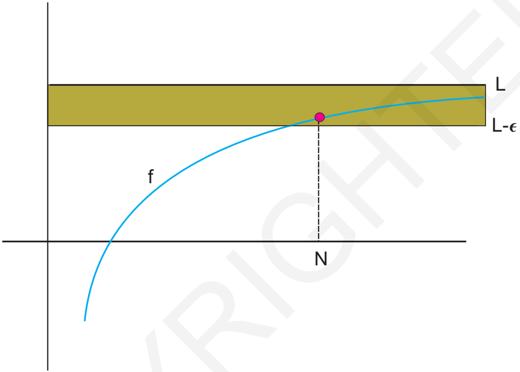
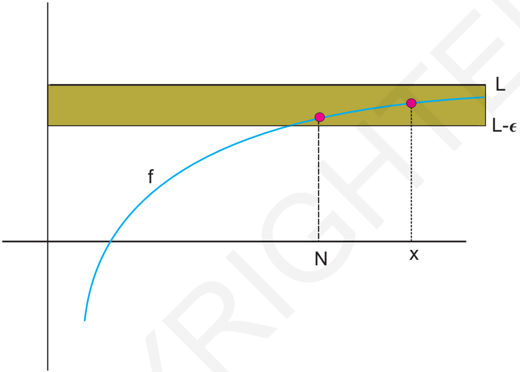


Proof:

1) Assume that S is bounded. Let $L = lub(S)$. Let $\epsilon > 0$. Then $L - \epsilon$ is not an upper bound for S so there exists $N \in [a, \infty)$ with

$$L - \epsilon < f(N) \leq L$$

Monotone Convergence Theorem for Functions



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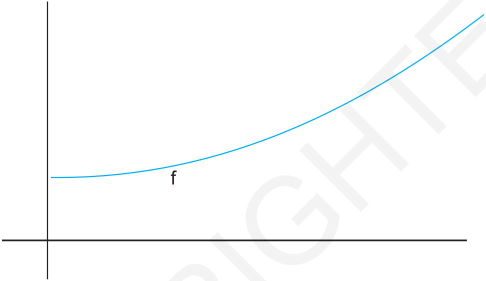
$$L - \epsilon < f(N) \leq L$$

Since f is non-decreasing, if $x \geq N$, then

$$L - \epsilon < f(N) \leq f(x) \leq L$$

Monotone Convergence Theorem for Functions



Proof:

2) Assume that S is not bounded above.

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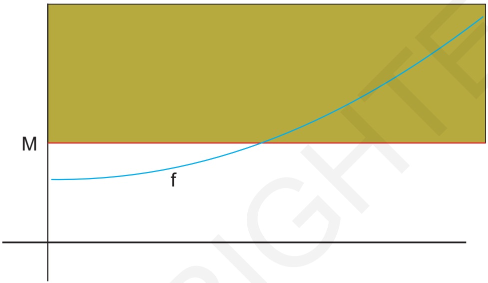

Proof:

2) Assume that S is not bounded above. Let $M > 0$.

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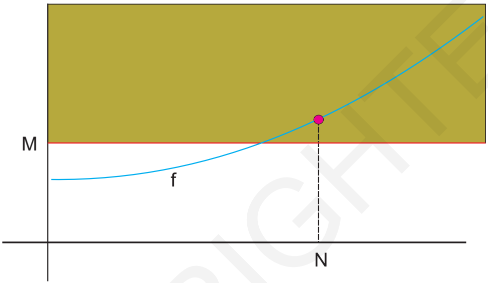
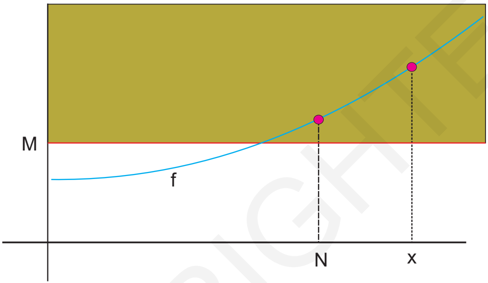


Proof:

2) Assume that S is not bounded above. Let $M > 0$. Since M is not an upper bound for S, there exists $N \in [a, \infty)$ with

$$f(N) > M$$

Monotone Convergence Theorem for Functions



Proof:

2) Assume that S is not bounded above. Let $M > 0$. Since M is not an upper bound for S, there exists $N \in [a, \infty)$ with

$$f(N) > M$$

Since f is non-decreasing, if $x \geq N$, then

$$f(x) \geq f(N) > M$$

Monotone Convergence Theorem for Functions

Application to Improper Integrals:

Assume that f is continuous and positive on $[a, \infty)$. For each $b \in [a, \infty)$ define

$$F(b) = \int_a^b f(t)\, dt.$$

1) Since F is increasing on $[a, \infty)$, we get that

$$\int_a^{\infty} f(t)\, dt$$

converges if and only if

$$S = \{F(b) \mid b \in [a, \infty)\}$$

is bounded above.

In case of convergence

$$\int_a^{\infty} f(t)\, dt = L = lub(S)$$

2) We have

$$\int_a^{\infty} f(t)\, dt = \lim_{n\to\infty} \int_a^n f(t)\, dt$$

where in this case $n \in \mathbb{N}$.

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Example: Observe that

$$F(b) = \int_0^b \cos(x)\,dx = \sin(x)\Big|_0^b = \sin(b)$$

Hence,

$$S = \{F(b) \mid b \in [0, \infty)\}$$

is bounded, but

$$\int_0^\infty \cos(x)\,dx$$

diverges.